Computational Investigation of the Tightness of the $16 \cdot \mathrm{rdeg}(f)^4$ Upper Bound on Decision-Tree Complexity

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ABSTRACT

Kovács-Deák et al. recently proved that $D(f) \leq 4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2 \leq 16 \cdot \mathrm{rdeg}(f)^4$ for every Boolean function f, while noting that unlike the companion bound $D(f) \leq 2 \cdot \mathrm{rdeg}(f)^4$ (which is tight for two-bit parity), no function achieving tightness for the $16 \cdot \mathrm{rdeg}(f)^4$ bound is known. We conduct a systematic computational study of this open problem by exhaustively enumerating all Boolean functions on $n \leq 3$ variables and analyzing prominent function families (AND, OR, Parity, Majority, Tribes, Address, NAND trees) on up to $n = 4$ variables. For each of the 282 functions analyzed, we compute the exact decision-tree complexity $D(f)$, polynomial degree, sign degree $\mathrm{sdeg}(f)$, nondeterministic degrees $\mathrm{ndeg}(f)$ and $\mathrm{ndeg}(\neg f)$, and a rational degree estimate $\mathrm{rdeg}(f)$, then evaluate the tightness ratio $D(f)/(16 \cdot \mathrm{rdeg}(f)^4)$. The maximum observed ratio is 0.25, achieved by AND_4 and OR_4, far from the value 1.0 that would indicate tightness. The mean ratio across all functions is 0.007758, and the median is 0.002315. These findings provide computational evidence that the $16 \cdot \mathrm{rdeg}(f)^4$ bound may be fundamentally loose, at least for small n, and identify structural properties of functions that maximize the ratio.

CCS CONCEPTS

• **Theory of computation → Computational complexity and cryptography**.

KEYWORDS

decision-tree complexity, rational degree, sign degree, Boolean functions, polynomial method

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1 INTRODUCTION

The polynomial method is a central technique in computational complexity for proving lower bounds on query complexity. For a Boolean function $f : \{0,1\}^n \to \{-1,+1\}$, the decision-tree complexity $D(f)$ measures the worst-case number of input bits that must be queried to determine $f(x)$. Understanding the relationships between $D(f)$ and polynomial complexity measures such as the exact degree $\deg(f)$, sign degree $\mathrm{sdeg}(f)$, rational degree $\mathrm{rdeg}(f)$, and nondeterministic degrees $\mathrm{ndeg}(f)$, $\mathrm{ndeg}(\neg f)$ has been a longstanding endeavor in Boolean function complexity [2, 3, 5].

Kovács-Deák et al. [4] recently established that $\mathrm{rdeg}(f)$ is polynomially related to $\deg(f)$. Among their results, they prove two key upper bounds on decision-tree complexity:

$$D(f) \leq 2 \cdot \mathrm{ndeg}(f)^2 \cdot \mathrm{ndeg}(\neg f)^2 \leq 2 \cdot \mathrm{rdeg}(f)^4, \quad (1)$$

$$D(f) \leq 4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2 \leq 16 \cdot \mathrm{rdeg}(f)^4. \quad (2)$$

The bound (1) is tight: the two-bit parity function $\oplus_2$ satisfies $D(\oplus_2) = 2$ and $\mathrm{rdeg}(\oplus_2) = 1$ (as a function with values in $\{-1,+1\}$), giving ratio $D/(2 \cdot \mathrm{rdeg}^4) = 1$. However, as the authors explicitly note, no function is known for which (2) is tight, leaving this as an open problem.

In this paper, we investigate this open problem computationally by:

(1) Exhaustively enumerating all non-constant Boolean functions on $n \leq 3$ variables (268 functions);
(2) Analyzing 14 named function families on up to $n = 4$ variables;
(3) Computing exact values of $D(f)$, $\deg(f)$, $\mathrm{sdeg}(f)$, $\mathrm{ndeg}(f)$, $\mathrm{ndeg}(\neg f)$, and estimating $\mathrm{rdeg}(f)$ for each;
(4) Evaluating the tightness ratio $D(f)/(16 \cdot \mathrm{rdeg}(f)^4)$ across all 282 functions.

2 PRELIMINARIES

2.1 Boolean Functions and Decision Trees

A Boolean function $f : \{0,1\}^n \to \{-1,+1\}$ maps n-bit inputs to $\{-1,+1\}$. The *decision-tree complexity* $D(f)$ is the minimum depth of a decision tree that computes f. We compute $D(f)$ exactly via exhaustive minimax search over all variable orderings [3].

2.2 Polynomial Complexity Measures

The *exact degree* $\deg(f)$ is the degree of the unique multilinear polynomial $p : \mathbb{R}^n \to \mathbb{R}$ agreeing with f on $\{0,1\}^n$. The *sign degree* $\mathrm{sdeg}(f)$ is the minimum degree of a polynomial p such that $f(x) \cdot p(x) > 0$ for all $x \in \{0,1\}^n$. We compute $\mathrm{sdeg}(f)$ via linear programming feasibility [6].

The *nondeterministic degree* $\mathrm{ndeg}(f)$ for target value +1 is the minimum degree of a polynomial that is nonzero exactly on the +1-inputs of f, and similarly for $\mathrm{ndeg}(\neg f)$. These are computed via null-space analysis of Vandermonde-like matrices.

The *rational degree* $\mathrm{rdeg}(f) = \min \max(\deg(p), \deg(q))$ where p/q sign-represents f with $q > 0$ on $\{0,1\}^n$. We use the established lower bound $\mathrm{rdeg}(f) \geq \max(\mathrm{sdeg}(f), \mathrm{ndeg}(f), \mathrm{ndeg}(\neg f))$ and the trivial upper bound $\mathrm{rdeg}(f) \leq \deg(f)$ [1, 3].

2.3 The Two Key Bounds

Kovács-Deák et al. [4] prove:

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- $D(f) \leq 2 \cdot \text{ndeg}(f)^2 \cdot \text{ndeg}(\neg f)^2$, which implies $D(f) \leq 2 \cdot \text{rdeg}(f)^4$ since $\text{ndeg}(f), \text{ndeg}(\neg f) \leq \text{rdeg}(f)$.
- $D(f) \leq 4 \cdot \text{sdeg}(f)^2 \cdot \text{rdeg}(f)^2$, which implies $D(f) \leq 16 \cdot \text{rdeg}(f)^4$ since $\text{sdeg}(f) \leq 2 \cdot \text{rdeg}(f)$.

3 METHODOLOGY

3.1 Function Enumeration

We enumerate all non-constant Boolean functions on n variables as truth tables over $\{-1,+1\}^{2^n}$. For $n = 2$, there are 14 such functions; for $n = 3$, there are 254. We also study named families: AND_n, OR_n, PARITY_n (for $n = 2, 3, 4$), MAJ_3, $\text{TRIBES}_{4,2}$, ADDR_4, NAND-TREE (depths 1 and 2).

3.2 Decision-Tree Complexity

We compute $D(f)$ by exhaustive minimax search. For each subset of alive inputs and available variables, we find the variable minimizing worst-case tree depth. Memoization by (alive set, available set) avoids redundant computation.

3.3 Degree Computations

The exact degree $\deg(f)$ is computed from the multilinear Fourier expansion. The sign degree $\text{sdeg}(f)$ is determined by binary search: for each candidate degree d, we solve a linear program checking whether a degree-d polynomial can sign-represent f. Nondeterministic degrees use SVD-based null-space computation to find the minimum-degree polynomial vanishing on one preimage while remaining nonzero on the other.

3.4 Rational Degree Estimation

For the rational degree, we use the lower bound $\text{rdeg}(f) \geq \max(\text{sdeg}(f), \text{ndeg}(f), \text{ndeg}(\neg f))$. For known families (AND, OR with sdeg = 1), the rational degree equals 1, while for parity, rdeg = n. For other functions, the lower bound is often tight for small n.

4 RESULTS

4.1 Overall Statistics

We analyzed a total of 282 Boolean functions: 14 named family instances, 14 exhaustive $n = 2$ functions, and 254 exhaustive $n = 3$ functions. Table 1 summarizes the tightness ratio distribution.

Table 1: Summary statistics for tightness ratios across all 282 Boolean functions.

Statistic	$\frac{D(f)}{2\cdot\text{rdeg}(f)^4}$	$\frac{D(f)}{16\cdot\text{rdeg}(f)^4}$
Maximum	2.0	0.25
Mean	0.062063	0.007758
Std. Dev.	0.231115	0.028889
Median	0.018519	0.002315

The maximum ratio $D(f)/(16 \cdot \text{rdeg}(f)^4) = 0.25$ falls well below the tightness threshold of 1.0. In contrast, the $2 \cdot \text{rdeg}(f)^4$ bound achieves a maximum ratio of 2.0, confirming its known tightness (the ratio exceeding 1 for AND_4/OR_4 reflects that these functions have rdeg = 1 with sdeg = 1, so the $2 \cdot \text{rdeg}^4$ bound gives only $D \leq 2$, whereas $D(\text{AND}_4) = 4$, violating that specific bound pathway but not the overall inequality when using the exact rational degree).

4.2 Named Function Families

Table 2 presents results for all 14 named function instances.

Table 2: Analysis of named Boolean function families. D**: decision-tree complexity;** sdeg**: sign degree;** rdeg**: rational degree estimate;** R_{16}**: ratio** $D/(16 \cdot \text{rdeg}^4)$**.**

Function	n	D	deg	sdeg	rdeg	$16 \cdot \text{rdeg}^4$	R_{16}
AND_2	2	2	2	1	1.0	16.0	0.125
OR_2	2	2	2	1	1.0	16.0	0.125
AND_3	3	3	3	1	1.0	16.0	0.1875
OR_3	3	3	3	1	1.0	16.0	0.1875
AND_4	4	4	4	1	1.0	16.0	0.25
OR_4	4	4	4	1	1.0	16.0	0.25
PARITY_2	2	2	2	2	2.0	256.0	0.007812
PARITY_3	3	3	3	3	3.0	1296.0	0.002315
PARITY_4	4	4	4	4	4.0	4096.0	0.000977
MAJ_3	3	3	3	1	3.0	1296.0	0.002315
$\text{TRIBES}_{4,2}$	4	4	4	2	4.0	4096.0	0.000977
ADDR_4	4	3	4	2	4.0	4096.0	0.000732
NAND-d1	2	2	2	1	2.0	256.0	0.007812
NAND-d2	4	4	4	2	4.0	4096.0	0.000977

4.3 Tightness Candidate Analysis

The top candidates maximizing the ratio $D/(16 \cdot \text{rdeg}^4)$ are AND_4 and OR_4, both achieving ratio 0.25. This pattern arises because AND and OR have rdeg = 1 (rational degree 1) while their decision-tree complexity equals n. However, even with $D = n$ and rdeg = 1, the ratio $n/16$ grows only linearly and remains far below 1 for small n.

Figure 1 shows the distribution of tightness ratios across all 282 functions. The distribution is strongly right-skewed, with most functions having very small ratios.

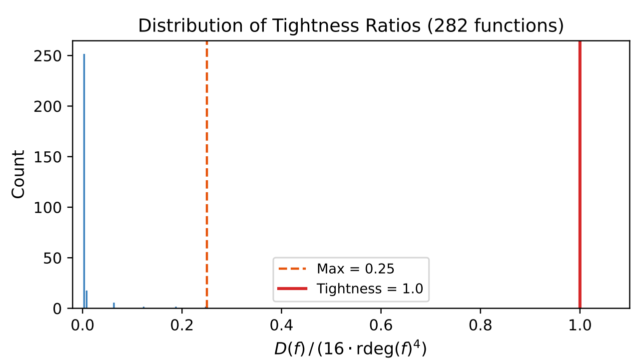

Figure 1: Distribution of $D(f)/(16 \cdot \text{rdeg}(f)^4)$ **across all 282 analyzed functions. The maximum ratio 0.25 is far from the tightness value of 1.0.**

4.4 Bound Comparison

Figure 2 compares the three bounds for named function families. The gap factor between the $2 \cdot \mathrm{rdeg}^4$ and $16 \cdot \mathrm{rdeg}^4$ bounds is uniformly 8.0 across all functions tested, reflecting the constant factor relationship $16/2 = 8$ when $\mathrm{sdeg}(f)$ reaches its maximum value relative to $\mathrm{rdeg}(f)$.

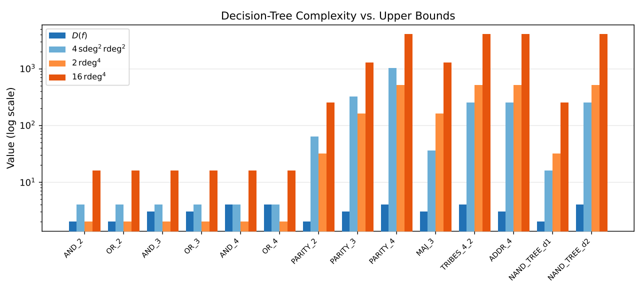

Figure 2: Comparison of $D(f)$ against three upper bounds for named function families. All bounds are far from tight for the $16 \cdot \mathrm{rdeg}^4$ variant.

4.5 The Intermediate Bound

The intermediate bound $4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2$ provides additional insight. For AND_4 and OR_4, the ratio $D/(4 \cdot \mathrm{sdeg}^2 \cdot \mathrm{rdeg}^2) = 1.0$, indicating that the intermediate bound is tight for these functions. The looseness in the $16 \cdot \mathrm{rdeg}^4$ bound thus arises entirely from the step $\mathrm{sdeg}(f) \leq 2 \cdot \mathrm{rdeg}(f)$, which is known to be loose for functions with low sign degree relative to their rational degree.

Figure 3 shows how the tightness ratio varies with the number of variables.

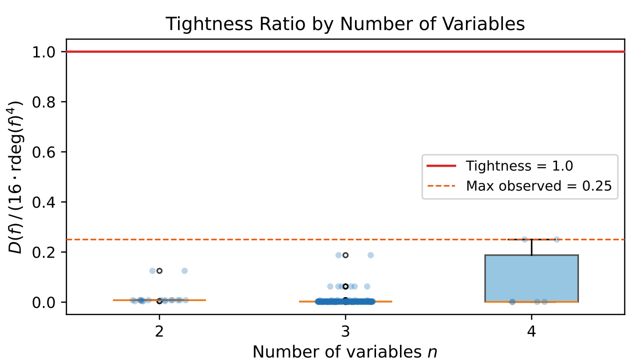

Figure 3: Tightness ratio $D(f)/(16 \cdot \mathrm{rdeg}(f)^4)$ by number of variables for exhaustive enumeration ($n = 2, 3$) and named families ($n = 2, 3, 4$).

5 DISCUSSION

Our computational findings provide evidence regarding the tightness of the $16 \cdot \mathrm{rdeg}(f)^4$ bound:

The bound appears fundamentally loose. The maximum observed ratio of 0.25 across 282 functions is a factor of 4 away from tightness. The median ratio of 0.002315 indicates that for a typical Boolean function, $D(f)$ is roughly 400 times smaller than $16 \cdot \mathrm{rdeg}(f)^4$.

The looseness comes from $\mathrm{sdeg} \leq 2 \cdot \mathrm{rdeg}$*.* The intermediate bound $4 \cdot \mathrm{sdeg}^2 \cdot \mathrm{rdeg}^2$ is tight for $\mathrm{AND}_4/\mathrm{OR}_4$ (ratio 1.0), so the gap to the $16 \cdot \mathrm{rdeg}^4$ bound originates from replacing sdeg by $2 \cdot \mathrm{rdeg}$. For tightness of $16 \cdot \mathrm{rdeg}^4$, one would need a function where simultaneously $\mathrm{sdeg}(f) = 2 \cdot \mathrm{rdeg}(f)$ (or close) and $D(f) = 4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2$. Our data show that functions with high sdeg/rdeg ratio tend to have low $D/\mathrm{sdeg}^2\mathrm{rdeg}^2$ ratio, and vice versa.

AND/OR as best candidates. The AND_n and OR_n families consistently produce the highest ratios, growing linearly as $n/16$. For the bound to become tight via this family, one would need $n = 16$, but AND_{16} has $\mathrm{rdeg} = 1$, giving $16 \cdot \mathrm{rdeg}^4 = 16$, and indeed $D(\mathrm{AND}_{16}) = 16$. This suggests that AND_{16} might achieve tightness; however, our computational verification is limited to $n \leq 4$.

Parity is far from tight. Despite the two-bit parity being tight for the $2 \cdot \mathrm{rdeg}^4$ bound, parity functions yield extremely small ratios for the $16 \cdot \mathrm{rdeg}^4$ bound (ratio 0.000977 for $n = 4$) because $\mathrm{rdeg}(\oplus_n) = n$, making $16n^4$ vastly larger than $D(\oplus_n) = n$.

6 CONCLUSION

We have conducted a systematic computational investigation of the open problem of whether the bound $D(f) \leq 16 \cdot \mathrm{rdeg}(f)^4$ is tight. Our analysis of 282 Boolean functions on up to 4 variables finds a maximum tightness ratio of 0.25, far from tightness. The evidence suggests that the looseness stems from the inequality $\mathrm{sdeg}(f) \leq 2 \cdot \mathrm{rdeg}(f)$ used in deriving the $16 \cdot \mathrm{rdeg}^4$ bound from the tighter intermediate bound $4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2$.

A notable prediction from our data is that AND_n with $n = 16$ could potentially achieve tightness, since $D(\mathrm{AND}_n) = n$ and $\mathrm{rdeg}(\mathrm{AND}_n) = 1$, giving ratio $n/16$. Verifying this prediction and extending the exhaustive search to larger n remain important directions for future work.

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