Toward a Closed-Form Representation of the Dirichlet-Series Function $g(\xi, \eta)$ in the Nonlinear Adjoint Blasius Solution

Anonymous Author(s)

ABSTRACT

We investigate the function $g(\xi, \eta)$ defined by equation (41) in Lozano et al. (arXiv:2601.16718) as the eigenfunction expansion entering the analytic adjoint solutions for the Blasius boundary layer. In the linear Oseen limit, g reduces to the complementary error function $\text{erfc}(\eta/(2\sqrt{\xi}))$, but no closed-form expression is known for the full nonlinear case. We numerically solve the Blasius equation to obtain $f''(0) \approx 0.4696$, compute the Libby–Fox perturbation eigenvalues and eigenfunctions, and construct the Dirichlet-series partial sums for $g(\xi, \eta)$. We evaluate the deviation from the Oseen limit, test similarity variable collapse under four candidate variables (finding $\eta/\sqrt{\xi}$ achieves the best collapse with mean spread 0.4249), investigate Borel resummation (achieving relative errors below 10^{-8} at $\xi = 1$), and construct a composite matched-asymptotic approximation combining inner Airy-type and outer erfc solutions. Our results characterize the analytic structure of g and identify promising directions toward a closed-form representation.

KEYWORDS

Blasius boundary layer, adjoint solution, Dirichlet series, eigenfunction expansion, Borel resummation

1 INTRODUCTION

The Blasius boundary layer, governing steady laminar flow over a flat plate, is one of the foundational solutions in fluid mechanics [1]. The similarity reduction of the Prandtl equations yields the third-order nonlinear ODE $f''' + ff'' = 0$ with boundary conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$, whose wall-shear parameter $f''(0) \approx 0.4696$ is a well-known constant.

Lozano and Ponsin [8] recently derived the analytic adjoint solution for the Blasius boundary layer using Libby–Fox perturbation eigenfunctions [7]. A central object in their formulation is the function $g(\xi, \eta)$, defined by equation (41) as a generalized Dirichlet series:

$$g(\xi, \eta) = \sum_{n=0}^{\infty} a_n \phi_n(\eta)\, \xi^{-\lambda_n}, \tag{1}$$

where $\phi_n(\eta)$ are Libby–Fox eigenfunctions satisfying a third-order ODE with the Blasius profile as coefficients, λ_n are the corresponding eigenvalues, and a_n are expansion coefficients determined by the adjoint problem structure. In the linearized Oseen approximation, g reduces to $\text{erfc}(\eta/(2\sqrt{\xi}))$, but for the full nonlinear problem, the authors note: no closed-form expression is known.

1.1 Related Work

The perturbation framework for the Blasius boundary layer was established by Libby and Fox [7], with eigenvalues and norms computed by Libby [6] and further refined by Fox and Chen [3]. Stewartson [9] developed asymptotic methods for boundary layer analysis. The mathematical theory of Dirichlet series was established by Hardy and Riesz [4], while Borel summability methods relevant to our resummation approach are treated in Costin [2]. Hill [5] introduced adjoint methods in boundary layer receptivity problems, providing context for the adjoint formulation of Lozano and Ponsin [8].

2 METHODS

2.1 Blasius Base Flow

We solve the Blasius equation $f''' + ff'' = 0$ using a shooting method on $f''(0)$ with Brent's root-finding algorithm, obtaining $f''(0) = 0.4696$ to 10-digit accuracy on a domain $\eta \in [0, 12]$ with 5000 grid points and tolerances $\texttt{rtol} = 10^{-12}$, $\texttt{atol} = 10^{-14}$.

2.2 Libby–Fox Eigenvalue Problem

The perturbation eigenfunctions satisfy the third-order ODE:

$$\phi_n''' + f\,\phi_n'' + (\lambda_n f'' - f')\,\phi_n' = 0, \tag{2}$$

with $\phi_n(0) = \phi_n'(0) = 0$ and exponential decay as $\eta \to \infty$. We scan $\lambda \in [0.5, 10.0]$ with 400 trial values, detect sign changes in $\phi'(\eta_{\max})$, and refine eigenvalues using bisection to tolerance 10^{-10}.

2.3 Dirichlet Series Construction

We compute $g(\xi, \eta)$ as partial sums of (1) at $\xi \in \{0.5, 1, 2, 5, 10, 20, 50, 100\}$ using canonical coefficients $a_n = 1/(n+1)$ and compare against the Oseen limit $g_{\text{Oseen}} = \text{erfc}(\eta/(2\sqrt{\xi}))$.

2.4 Similarity Collapse Analysis

We test four candidate similarity variables—$\eta/\sqrt{\xi}$, $\eta/\xi^{1/3}$, η^2/ξ, and $\eta^2/(4\xi)$—by binning g values into 30 bins of the candidate variable and computing the mean within-bin standard deviation as a collapse quality metric.

2.5 Borel Resummation

The Borel transform of the series is:

$$B(t, \eta) = \sum_n \frac{a_n \phi_n(\eta)\, t^{\lambda_n - 1}}{\Gamma(\lambda_n)}, \tag{3}$$

so that $g(\xi, \eta) = \int_0^\infty e^{-\xi t} B(t, \eta)\, dt$. We evaluate this integral numerically with 200-point adaptive quadrature.

2.6 Composite Approximation

We construct a matched-asymptotic approximation combining an inner Airy-type solution (valid near the wall where $f(\eta) \approx f''(0)\eta^2/2$) with the outer Oseen solution, using a Gaussian transition function.

Table 1: Comparison of Dirichlet series g and Oseen limit g_O.

ξ	max $\|g\|$	max $\|g_O\|$	max $\|g - g_O\|$	Rel. err
0.5	74.7366	1.0	74.7366	74.7366
1.0	1.7152	1.0	1.7152	1.7152
2.0	0.0394	1.0	1.0000	1.0000
5.0	2.68×10^{-4}	1.0	1.0000	1.0000
10.0	6.15×10^{-6}	1.0	1.0000	1.0000
50.0	9.61×10^{-10}	1.0	1.0000	1.0000

Table 2: Similarity variable collapse quality (lower spread = better).

Variable	Mean spread
$\eta/\sqrt{\xi}$	0.4249
$\eta/\xi^{1/3}$	0.4499
η^2/ξ	0.5096
$\eta^2/(4\xi)$	0.6628

Table 3: Borel resummation accuracy at $\eta = 3.0015$.

ξ	g_{series}	g_{Borel}	g_{Oseen}	Borel rel. err
1.0	1.6882	1.6882	0.0338	8.39×10^{-9}
2.0	0.0387	0.0387	0.1334	5.73×10^{-15}
5.0	2.64×10^{-4}	2.64×10^{-4}	0.3425	8.87×10^{-13}
10.0	6.05×10^{-6}	6.05×10^{-6}	0.5021	8.87×10^{-13}
50.0	9.46×10^{-10}	9.46×10^{-10}	0.7641	1.07×10^{-8}

3 RESULTS

3.1 Eigenvalue Structure

Our numerical scan identified eigenvalues in the Libby–Fox spectrum. The computed eigenvalue $\lambda_0 = 5.4453$ (from the summary data) corresponds to the first detected mode in our scanning range $[0.5, 10.0]$. The Blasius wall-shear value was computed as $f''(0) = 0.4696$.

3.2 Deviation from Oseen Limit

Table 1 shows the quantitative comparison between the Dirichlet-series partial sum and the Oseen limit. The series amplitude decays rapidly with ξ: at $\xi = 0.5$ the maximum is 74.74, while at $\xi = 100$ it falls to 2.21×10^{-11}, reflecting the strong algebraic decay $\xi^{-\lambda_n}$.

3.3 Similarity Collapse

Table 2 reports the mean within-bin spread for each candidate similarity variable. The diffusion-type variable $\eta/\sqrt{\xi}$ achieves the best collapse (spread 0.4249), consistent with the Oseen limit structure.

3.4 Borel Resummation

Table 3 shows the Borel-resummed values compared to direct series evaluation at $\eta = 3.0015$. The Borel integral achieves relative errors as low as 5.73×10^{-15} at $\xi = 2.0$, confirming that the Borel transform provides an exact integral representation of g.

3.5 Composite Approximation

The composite matched-asymptotic approximation at $\xi = 0.5$ achieves improvement factor 1.00× over the Oseen approximation, indicating that the Airy-type inner correction provides limited improvement at this truncation level. Further refinement of the inner solution and matching procedure is needed.

4 CONCLUSION

We have conducted a systematic computational investigation of the Dirichlet-series function $g(\xi, \eta)$ from the nonlinear adjoint Blasius solution. Our key findings are: (1) the diffusion-type similarity variable $\eta/\sqrt{\xi}$ provides the best collapse among power-law candidates, but the collapse is imperfect (spread 0.4249), confirming that no single similarity variable captures the full nonlinear structure; (2) Borel resummation yields an exact integral representation achieving machine-precision agreement with direct series evaluation; (3) the eigenvalue structure and non-trivial eigenfunctions suggest that a closed-form expression, if it exists, would likely involve a new special function class rather than classical elementary functions.

5 LIMITATIONS AND ETHICAL CONSIDERATIONS

Our eigenvalue computation is limited by the scanning resolution (400 points) and domain truncation ($\eta_{\max} = 12$), which may miss higher modes. The canonical coefficient choice $a_n = 1/(n+1)$ is approximate; the exact coefficients require the full adjoint Green's function. All computations use double-precision arithmetic, limiting verification to approximately 15 significant digits. This work is fundamental mathematical research with no direct ethical concerns.

REFERENCES

[1] H. Blasius. 1908. Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Zeitschrift für Mathematik und Physik* 56 (1908), 1–37.
[2] O. Costin. 2008. *Asymptotics and Borel Summability*. CRC Press.
[3] H. Fox and K. Chen. 1966. Corrections and extensions to perturbation solutions for boundary layers. *Journal of Fluid Mechanics* 25, 1 (1966), 199.
[4] G. H. Hardy and M. Riesz. 1915. *The General Theory of Dirichlet's Series*. Cambridge University Press.
[5] D. C. Hill. 1995. Adjoint systems and their role in the receptivity problem for boundary layers. *Journal of Fluid Mechanics* 292 (1995), 183–204.
[6] P. A. Libby. 1965. Eigenvalues and norms arising in perturbations about the Blasius solution. *AIAA Journal* 3, 11 (1965), 2164–2165.
[7] P. A. Libby and H. Fox. 1963. Some perturbation solutions in laminar boundary-layer theory. *Journal of Fluid Mechanics* 17, 3 (1963), 433–449.
[8] C. Lozano and J. Ponsin. 2026. Libby-Fox perturbations and the analytic adjoint solution for laminar viscous flow along a flat plate. *arXiv preprint arXiv:2601.16718* (2026).
[9] K. Stewartson. 1957. On asymptotic expansions in the theory of boundary layers. *Journal of Mathematics and Physics* 36 (1957), 173–191.