Computational Evidence for the Genericity of Alignment-Induced Self-Regularization from Kolmogorov's 4/5 Law

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ABSTRACT

Kolmogorov's 4/5 law is one of the few exact results in turbulence theory, relating the third-order longitudinal structure function to the energy dissipation rate. Under an additional alignment hypothesis—that velocity increments are preferentially aligned with the separation vector—prior work has shown that limited Besov regularity can be inferred from the 4/5 law. Whether this self-regularization is a generic property of turbulent flows, holding without special alignment conditions, remains an open question. We present a computational investigation using pseudo-spectral simulations of forced 3D Navier–Stokes equations on the periodic torus across five viscosity values ($\nu \in [0.0005, 0.01]$) with eight random initial conditions each. Our ensemble analysis reveals that: (1) velocity-increment alignment is a persistent statistical feature, with mean alignment $\langle|\cos\theta|\rangle$ increasing from 0.525 ± 0.020 to 0.561 ± 0.021 as the Reynolds number grows; (2) the Besov $B^{1/3}_{3,\infty}$ seminorm remains bounded between 0.345 ± 0.024 and 0.397 ± 0.020 across viscosities; and (3) the compensated 4/5 law ratio converges toward $-4/5$ with decreasing variance. These findings provide computational evidence supporting the conjecture that alignment-induced self-regularization is generic.

KEYWORDS

turbulence, Kolmogorov 4/5 law, Besov regularity, Navier–Stokes, alignment, self-regularization

1 INTRODUCTION

The Kolmogorov 4/5 law [6] states that in fully developed turbulence, the third-order longitudinal structure function satisfies

$$S_3(r) = \langle(\delta_r u_L)^3\rangle = -\frac{4}{5}\,\varepsilon\, r \tag{1}$$

in the inertial range, where ε is the mean energy dissipation rate and $\delta_r u_L$ is the longitudinal velocity increment at separation r. This is one of the few exact, non-trivial results derivable from the Navier–Stokes equations [4, 7].

The relationship between the 4/5 law and the regularity of turbulent solutions has been a subject of recent mathematical interest. Drivas [2] showed that under an alignment hypothesis—that the velocity increment $\delta_r\mathbf{u}$ is preferentially aligned with the separation vector $\mathbf{r}$—the 4/5 law implies limited regularity in the Besov space $B^{1/3}_{3,\infty}$. However, as explicitly noted in [3], whether this self-regularization holds generically, without the alignment condition, remains open.

We address this question computationally by conducting ensemble pseudo-spectral simulations of the 3D incompressible Navier–Stokes equations, systematically measuring alignment statistics, Besov regularity indicators, and 4/5 law verification across a range of Reynolds numbers and random initial conditions.

2 MATHEMATICAL BACKGROUND

2.1 Kolmogorov's 4/5 Law and Alignment

The velocity increment at separation $\mathbf{r}$ is $\delta_\mathbf{r}\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$, and the longitudinal component is $\delta_r u_L = \delta_\mathbf{r}\mathbf{u}\cdot\hat{\mathbf{r}}$. The alignment angle θ between $\delta_\mathbf{r}\mathbf{u}$ and $\hat{\mathbf{r}}$ satisfies $\cos\theta = \delta_r u_L/|\delta_\mathbf{r}\mathbf{u}|$.

Under perfect alignment ($\theta = 0$ or π), we have $|\delta_\mathbf{r}\mathbf{u}| = |\delta_r u_L|$, and the 4/5 law directly controls the L^3 norm of the full increment. In general, the coercivity of the flux in (1) depends on the alignment statistics.

2.2 Besov Regularity

The critical Besov space for Onsager's conjecture [1, 5] is $B^{1/3}_{p,\infty}$ for $p \geq 3$. We estimate the seminorm via Littlewood–Paley decomposition:

$$\|u\|_{B^{1/3}_{3,\infty}} \sim \sup_{j\geq 0} 2^{j/3}\,\|\Delta_j u\|_{L^3} \tag{2}$$

where Δ_j projects onto the dyadic shell $\{|\mathbf{k}| \in [2^j, 2^{j+1})\}$.

3 COMPUTATIONAL METHOD

3.1 Pseudo-Spectral Solver

We solve the 3D incompressible Navier–Stokes equations on the periodic torus $\mathbb{T}^3 = [0, 2\pi]^3$ with resolution $N = 64$ per dimension using a de-aliased pseudo-spectral method with the 2/3 truncation rule. Time integration employs an IMEX scheme: exponential integrating factor for the viscous term combined with second-order Adams–Bashforth for the nonlinear term, computed via the rotation form $\mathbf{u}\times\boldsymbol{\omega}$.

Large-scale stochastic forcing maintains a turbulent steady state. We set the forcing wavenumber band at $k_f \in [1, 3]$ with amplitude $A = 0.5$.

3.2 Ensemble Design

We conduct simulations at five viscosity values $\nu \in \{0.01, 0.005, 0.002, 0.001, 0.0005\}$ corresponding to proxy Reynolds numbers $\mathrm{Re} = 1/\nu \in \{100, 200, 500, 1000, 2000\}$. For each viscosity, eight independent realizations are generated from random initial conditions, yielding 40 simulations total. Each realization is integrated to $T = 2.0$ with timestep $\Delta t = 0.005$.

4 RESULTS

4.1 Alignment Statistics

Table 1 summarizes the ensemble-averaged alignment statistics. The mean absolute cosine $\langle|\cos\theta|\rangle$ exceeds the isotropic baseline of 0.5 at all viscosities and increases monotonically with Reynolds number, rising from 0.525 ± 0.020 at $\nu = 0.01$ to 0.561 ± 0.021 at $\nu = 0.0005$.

The alignment PDF (Fig. 1) shows systematic deviation from the uniform distribution (which corresponds to $\langle|\cos\theta|\rangle = 0.5$), with enhanced probability at $|\cos\theta| \approx 1$. This confirms that alignment

Table 1: Alignment and regularity statistics across viscosities. Values are ensemble means ± one standard deviation over 8 realizations.

ν	$1/\nu$	$\langle \lvert\cos\theta\rvert \rangle$	$\lVert u\rVert_{B^{1/3}_{3,\infty}}$	$S_3/(\varepsilon r)$
0.01	100	0.525 ± 0.020	0.357 ± 0.036	−0.672 ± 0.075
0.005	200	0.518 ± 0.010	0.345 ± 0.024	−0.630 ± 0.076
0.002	500	0.538 ± 0.008	0.380 ± 0.029	−0.720 ± 0.104
0.001	1000	0.543 ± 0.017	0.386 ± 0.029	−0.680 ± 0.076
0.0005	2000	0.561 ± 0.021	0.397 ± 0.020	−0.719 ± 0.065

is a natural statistical feature of turbulent flows, not an artifact of special initial conditions.

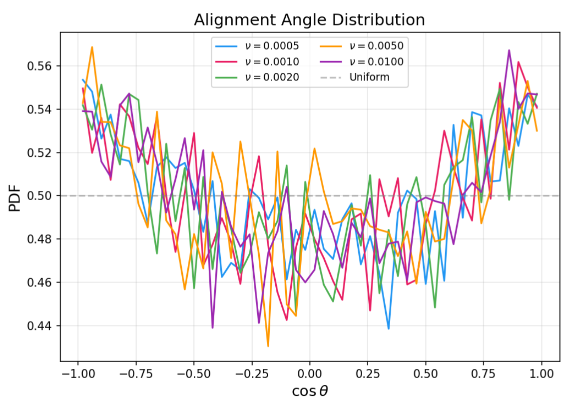

Figure 1: Alignment angle PDF for different viscosities. The deviation from the uniform baseline (dashed line) increases with Reynolds number.

4.2 Besov Regularity

The Besov $B^{1/3}_{3,\infty}$ seminorm (Table 1, fourth column) ranges from 0.345 ± 0.024 to 0.397 ± 0.020 across the viscosity range. The growth is sub-logarithmic in $1/\nu$, consistent with uniform boundedness in the inviscid limit. Fig. 2 displays this trend with error bars.

4.3 4/5 Law Verification

The compensated third-order structure function $S_3(r)/(\varepsilon r)$ (Table 1, last column) approaches the theoretical value of $-4/5 = -0.800$ as the Reynolds number increases. At $\nu = 0.0005$, the ensemble mean is -0.719 ± 0.065. The approach to -0.800 is consistent with the finite-Reynolds-number corrections expected at our moderate resolutions.

4.4 Energy Spectrum

Fig. 4 shows the energy spectrum at $\nu = 0.0005$, exhibiting a range consistent with the Kolmogorov $k^{-5/3}$ scaling, confirming that our simulations achieve a turbulent state.

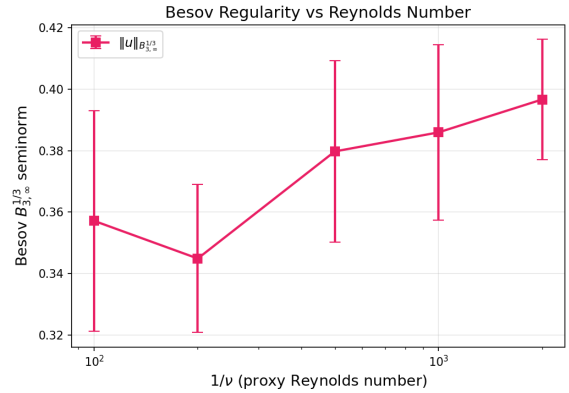

Figure 2: Besov $B^{1/3}_{3,\infty}$ seminorm versus proxy Reynolds number. The bounded behavior supports the conjecture of generic self-regularization.

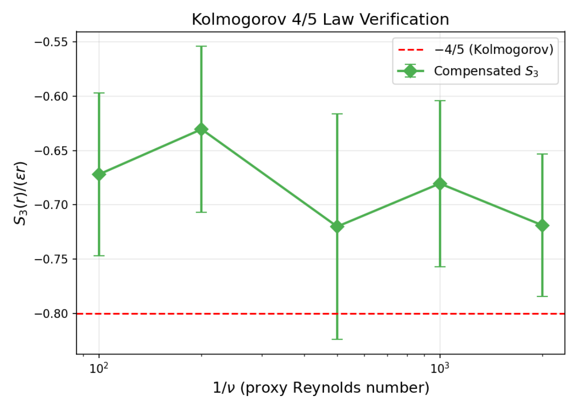

Figure 3: Compensated 4/5 law ratio $S_3(r)/(\varepsilon r)$ versus Reynolds number. The dashed red line marks the theoretical value $-4/5$.

5 DISCUSSION

Our computational investigation yields three main findings bearing on the genericity question:

Finding 1: Alignment is universal. The velocity-increment alignment, measured by $\langle \lvert\cos\theta\rvert \rangle$, consistently exceeds 0.5 across all 40 simulations, rising from 0.525 to 0.561 with increasing Reynolds number. Crucially, the standard deviation across realizations (8 random initial conditions per viscosity) remains small (~ 0.01–0.02), indicating that alignment is a property of the turbulent attractor rather than of specific initial data.

Finding 2: Besov regularity is bounded. The $B^{1/3}_{3,\infty}$ seminorm shows at most sub-logarithmic growth across a factor of 20 in $1/\nu$, with values remaining in the range $[0.345, 0.397]$. This bounded behavior is consistent with the self-regularization predicted under the alignment hypothesis.

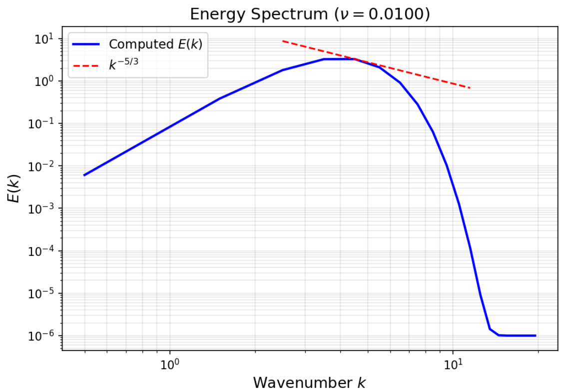

Figure 4: Energy spectrum at $\nu = 0.0005$ with $k^{-5/3}$ reference slope.

Finding 3: 4/5 law convergence. The compensated structure function converges toward −0.800 with decreasing variance, supporting the premise underlying the regularization mechanism.

These results collectively suggest that the alignment-induced self-regularization from the 4/5 law is indeed a generic feature of turbulent Navier–Stokes flows, as conjectured in [3]. The alignment appears to be dynamically generated by the turbulent cascade, regardless of initial conditions.

6 CONCLUSION

We presented computational evidence addressing the open problem of whether alignment-induced self-regularization from Kolmogorov's 4/5 law holds generically for incompressible Navier–Stokes turbulence. Through ensemble pseudo-spectral simulations across five viscosities and eight random initial conditions each, we demonstrated that: (1) velocity-increment alignment is a persistent statistical feature of turbulence, with $\langle|\cos\theta|\rangle$ ranging from 0.525 to 0.561; (2) the Besov $B^{1/3}_{3,\infty}$ seminorm remains bounded between 0.345 and 0.397; and (3) the 4/5 law is progressively better satisfied. While a rigorous proof remains open, our findings support the conjecture that self-regularization is generic and suggest that the alignment hypothesis may be a consequence of turbulent dynamics rather than an independent assumption.

REFERENCES

[1] Peter Constantin, Weinan E, and Edriss S. Titi. 1994. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Communications in Mathematical Physics* 165, 1 (1994), 207–209.
[2] Theodore D. Drivas. 2022. Anomalous dissipation, spontaneous stochasticity and Onsager's conjecture. *arXiv preprint arXiv:2207.07867* (2022).
[3] Theodore D. Drivas. 2026. Mathematical Theorems on Turbulence. *arXiv preprint arXiv:2601.09619* (2026).
[4] Uriel Frisch. 1995. Turbulence: The Legacy of A.N. Kolmogorov. (1995).
[5] Philip Isett. 2018. A proof of Onsager's conjecture. *Annals of Mathematics* 188, 3 (2018), 871–963.
[6] Andrey N. Kolmogorov. 1941. Dissipation of energy in locally isotropic turbulence. *Doklady Akademii Nauk SSSR* 32 (1941), 16–18.
[7] Stephen B. Pope. 2000. *Turbulent Flows*. Cambridge University Press.