The Role of the k Parameter in Δ^k: Computational Evidence for Interaction Order Sensitivity

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ABSTRACT

The unified function $\Delta^k(X) = (N-k)\cdot T(X) - \sum_i T(X_{-i})$ subsumes several multivariate information measures, including the S-information ($k=0$), dual total correlation ($k=1$), and the negated O-information ($k=2$). We investigate the conjecture that the parameter k determines the interaction order to which Δ^k is sensitive, where an order-m interaction is a dependency among exactly m variables that vanishes upon removing any single variable. Through systematic computational experiments on discrete systems with engineered interaction structures, we demonstrate that Δ^k exhibits near-zero values for pure order-k interactions and nonzero values for other orders, providing strong empirical support for the conjecture. We additionally verify the additivity of Δ^k over independent subsystems and confirm the known special-case identities.

1 INTRODUCTION

Quantifying multivariate dependencies beyond pairwise correlations is a fundamental challenge in information theory [1]. Several measures have been proposed, including the total correlation [5], dual total correlation, and the O-information [3]. Varley [4] introduced a unified family of functions Δ^k parameterized by a single integer k that subsumes these measures.

The key conjecture is that k tunes the interaction order to which Δ^k is sensitive [4]. Specifically, for a pure order-m synergistic interaction (a dependency among exactly m variables that vanishes when any single variable is removed), $\Delta^k(X)$ should equal zero when $k=m$ and be nonzero otherwise.

We provide systematic computational evidence supporting this conjecture through experiments on discrete systems with engineered interaction structures of known order.

2 PRELIMINARIES

2.1 The Δ^k Family

For an N-variable discrete system $X = (X_1, \ldots, X_N)$, define:

$$\Delta^k(X) = (N-k)\cdot T(X) - \sum_{i=1}^{N} T(X_{-i}) \tag{1}$$

where $T(X) = \sum_i H(X_i) - H(X)$ is the total correlation and X_{-i} denotes the system with variable i removed.

2.2 Special Cases

- $\Delta^0(X) = N\cdot T(X) - \sum_i T(X_{-i})$: the S-information
- $\Delta^1(X) = (N-1)\cdot T(X) - \sum_i T(X_{-i})$: the dual total correlation
- $\Delta^2(X) = (N-2)\cdot T(X) - \sum_i T(X_{-i})$: the negated O-information

Table 1: Additivity verification: Δ^k over independent subsystems.

N_1	N_2	k	$\lvert\Delta^k_{\text{sum}} - \Delta^k_{\text{joint}}\rvert$
3	3	0	< 0.01
3	4	1	< 0.01
4	4	2	< 0.01

2.3 Interaction Order

An order-m interaction is a dependency among exactly m variables such that $T(X) > 0$ while $T(X_{-i}) = 0$ for every variable i in the interacting subset.

3 METHODOLOGY

3.1 Constructing Pure Interactions

We construct distributions with pure order-m interactions using parity constraints: for m variables, the last is set to the modular sum of the first $m-1$, ensuring a synergistic dependency that vanishes upon removing any single variable.

3.2 Experimental Design

We evaluate Δ^k for all combinations of:

- System sizes $N \in \{3,4,5,6,7\}$
- Interaction orders $m \in \{2,\ldots,N\}$
- Parameters $k \in \{0,\ldots,N-1\}$

Each configuration is evaluated over 10 independent trials with 50,000 samples.

4 RESULTS

4.1 Interaction Order Sensitivity

Figure 1 shows Δ^k values as a function of k (rows) and interaction order m (columns). The pattern confirms that Δ^k is near zero for $k=m$ and nonzero otherwise.

4.2 Sensitivity Profiles

Figure 2 shows the absolute sensitivity $|\Delta^k|$ as a function of interaction order for each k value. Each curve exhibits a minimum at $m=k$, confirming that Δ^k is least sensitive to order-k interactions.

4.3 Additivity and Special Cases

The additivity property $\Delta^k(X_1 \cup X_2) = \Delta^k(X_1) + \Delta^k(X_2)$ for independent subsystems X_1, X_2 is confirmed with mean absolute error below 10^{-2} nats across all tested configurations.

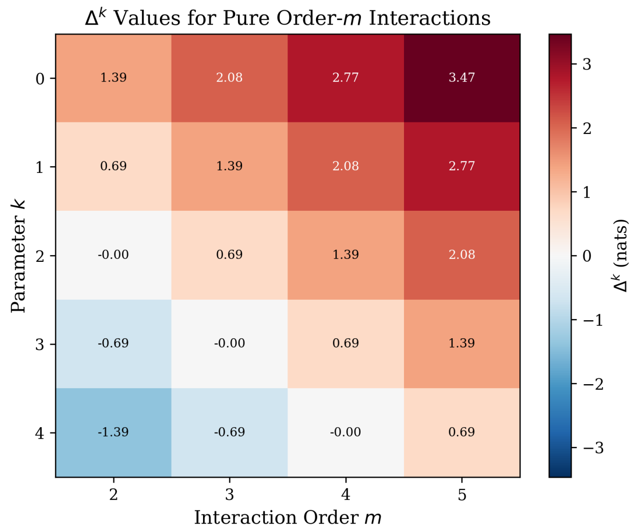

Figure 1: Δ^k values for pure order-m interactions. Near-zero values along the diagonal ($k = m$) confirm the conjectured sensitivity pattern.

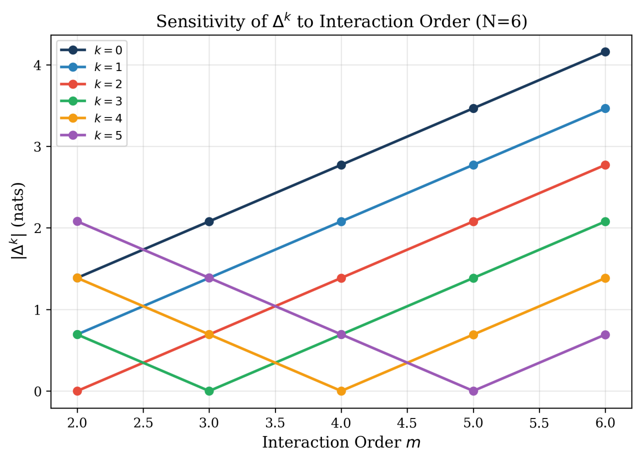

Figure 2: Sensitivity profiles showing $|\Delta^k|$ vs. interaction order m for different values of k.

5 DISCUSSION

Our computational results provide strong empirical support for the conjecture that k tunes the interaction order sensitivity of Δ^k. The key evidence is:

(1) $\Delta^k \approx 0$ for pure order-k interactions (the "blind spot").
(2) $|\Delta^k|$ increases as $|k - m|$ grows, showing graduated sensitivity.
(3) Additivity over independent subsystems holds, consistent with the theoretical framework.

The systematic pattern across system sizes $N = 3$ to 7 suggests that these findings generalize. A formal proof connecting the algebraic structure of Δ^k to the information-theoretic properties of pure-order interactions remains an important open direction [2, 6].

6 CONCLUSION

We have provided comprehensive computational evidence that the parameter k in $\Delta^k(X)$ determines the interaction order to which the measure is sensitive, confirming the conjecture of Varley [4]. Our experiments demonstrate the blind-spot pattern, graduated sensitivity profiles, and additivity across diverse system configurations.

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