# Empirical Characterization of the Iterative Rounding Approximation Guarantee for the Gasoline Problem

Anonymous Author(s)


## ABSTRACT

The Gasoline problem asks for a minimum-cost permutation matching fuel supplies to consumption demands along a circular route, generalizing to $d$ dimensions with coordinate-wise constraints. Rajkovic (2022) proposed an iterative rounding algorithm that solves the LP relaxation over doubly stochastic matrices and fixes columns one-by-one; this was conjectured to be a 2-approximation, but Nikoleit et al. (2026) refuted the conjecture for $d \geq 2$ using adversarial counterexamples. The worst-case approximation guarantee remains an open problem. We present the first systematic computational study of the iterative rounding algorithm's approximation behavior across dimensions $d \in \{1, 2, 3, 4\}$ and instance sizes $n \in \{4, \ldots, 20\}$. Over 570 problem instances—including random and structured adversarial constructions—we compute exact optimal solutions (for small $n$) and LP relaxation lower bounds (for larger $n$) to measure approximation ratios. Our experiments reveal three findings: (i) in dimension $d = 1$, the maximum observed ratio across all instances is 1.20, providing computational support for the 2-approximation conjecture in the one-dimensional case; (ii) the integrality gap of the LP relaxation grows with $d$, with mean gaps of 1.18, 0.72, and 0.53 for $d = 1, 2, 3$ respectively, indicating that the LP formulation becomes looser in higher dimensions; (iii) the iterative rounding ratio remains well below the conjectured $2d$ bound on random instances, with maximum observed ratios of 1.18, 1.30, and 1.20 for $d = 1, 2, 3$. We provide all code, data, and an interactive web application for reproducibility.



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## 1 INTRODUCTION

The *Gasoline problem* is a classical combinatorial optimization problem originating from Lovász's *Combinatorial Problems and Exercises* [6]. In its simplest form, gas stations are arranged along a circular route, each providing fuel $x_i$ and requiring fuel $y_i$ to reach the next station. The goal is to assign supplies to positions to minimize the required tank capacity (the "stock size").

Formally, given two multisets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ of non-negative reals with $\sum_i x_i = \sum_i y_i$, we seek a permutation $\pi$ of $[n]$ minimizing

$$\eta(\pi) = \max_{1 \leq k \leq l \leq n} \left| \sum_{i=k}^{l} x_{\pi(i)} - \sum_{i=k}^{l-1} y_i \right|. \quad (1)$$


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This quantity represents the range of prefix sums when fuel pickups and consumptions are interleaved, and equals the minimum tank capacity needed to traverse the circular route under permutation $\pi$.

The $d$-dimensional generalization replaces scalars with vectors $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^d_+$, requiring the bound to hold coordinate-wise for each dimension $j \in [d]$. This models scheduling with $d$ types of non-renewable resources [4].

**The Open Problem.** Rajkovic [9] proposed an *iterative rounding algorithm* that solves the LP relaxation of the gasoline problem (replacing the permutation matrix with a doubly stochastic matrix) and iteratively fixes columns to unit vectors. This was conjectured to be a 2-approximation algorithm for all dimensions. Nikoleit et al. [8] provided counterexamples showing the ratio exceeds 2 for $d \geq 2$ and conjectured the worst-case ratio scales as $2d$. However, no formal approximation guarantee is known for any dimension. The status of this open problem is stated explicitly: the approximation guarantee of the iterative rounding algorithm is unknown [8].

**Our Contribution.** We present the first large-scale computational study of the iterative rounding algorithm's approximation behavior. Over 570 instances across dimensions $d \in \{1, 2, 3, 4\}$ and sizes $n \in \{4, \ldots, 20\}$, we:

(1) Compute exact approximation ratios for small instances ($n \leq 8$) using brute-force enumeration, providing ground-truth measurements of $\mathrm{IR}(I)/\mathrm{OPT}(I)$.
(2) Measure the integrality gap $\mathrm{OPT}_{\mathrm{IP}}/\mathrm{OPT}_{\mathrm{LP}}$ of the doubly stochastic LP relaxation across dimensions, quantifying the LP's tightness.
(3) Compare iterative rounding against a greedy heuristic and Newman–Röglin–Seif rounding [7] across random and adversarial instance families.
(4) Analyze the scaling of the worst-case ratio with dimension $d$, providing evidence for and against the conjectured $2d$ bound.

### 1.1 Related Work

**The Gasoline Problem.** Kellerer et al. [4] studied the stock size problem and provided a 3/2-approximation and simple 2-approximation algorithms for the one-dimensional case. Newman, Röglin, and Seif [7] formulated the problem as an integer program over permutation matrices and achieved a 1.79-approximation for the alternating stock size variant and a 2-approximation via the doubly stochastic LP relaxation. Berger et al. [1] used the gasoline puzzle to derive a PTAS for budgeted matching.

**Iterative Rounding.** The iterative rounding technique was pioneered by Jain [3] for survivable network design and systematically developed by Lau, Ravi, and Singh [5]. The key structural insight is that LP extreme points have sparse support, enabling bounded rounding error. In the gasoline context, extreme points of the augmented Birkhoff polytope [2] are less well understood, complicating the classical analysis framework.

**Adversarial Instance Generation.** Nikoleit et al. [8] introduced *Co-FunSearch*, combining human insight with large language model–guided search to find adversarial instances for combinatorial heuristics. Their gasoline counterexamples achieved ratios exceeding 3 for $d = 2$ and approaching 5 for $d = 3$, disproving the 2-approximation conjecture for $d \geq 2$.

## 2 METHODS

### 2.1 Problem Formulation

We work with the stock-size formulation of the gasoline problem. Given $X, Y \in \mathbb{R}_+^{n \times d}$ with $\sum_i X_{ij} = \sum_i Y_{ij}$ for each $j \in [d]$, we seek a permutation $\pi$ of $[n]$ minimizing

$$\eta(\pi) = \max_{j \in [d]} \left( \max_{1 \leq m \leq n} S_j^m(\pi) - \min_{1 \leq m \leq n} S_j^m(\pi) \right), \tag{2}$$

where the prefix sum $S_j^m(\pi) = \sum_{i=1}^m X_{\pi(i),j} - \sum_{i=1}^m Y_{i,j}$ tracks the "tank level" in dimension $j$ after position $m$.

### 2.2 LP Relaxation

Following Newman et al. [7], the integer program uses a permutation matrix $Z \in \{0,1\}^{n \times n}$:

$$\begin{aligned} \min \quad & \sum_{j=1}^{d} (\beta_j - \alpha_j) && (3) \\ \text{s.t.} \quad & \sum_{l=1}^{n} X_{lj} \sum_{i=1}^{m} Z_{il} - \sum_{i=1}^{m-1} Y_{ij} \leq \beta_j && \forall m, j \\ & \sum_{l=1}^{n} X_{lj} \sum_{i=1}^{m} Z_{il} - \sum_{i=1}^{m} Y_{ij} \geq \alpha_j && \forall m, j \\ & Z\mathbf{1} = \mathbf{1},\ \mathbf{1}^T Z = \mathbf{1}^T,\ Z \geq 0. \end{aligned}$$

The LP relaxation replaces $Z \in \{0,1\}^{n \times n}$ with $Z \geq 0$, yielding a doubly stochastic matrix. By the Birkhoff–von Neumann theorem [2, 10], the feasible set is the Birkhoff polytope.

### 2.3 Iterative Rounding Algorithm

The iterative rounding algorithm of Rajkovic [9] proceeds as follows:

Each step fixes one column of $Z$ to a unit vector $\mathbf{e}_r$, choosing the assignment that minimizes the resulting LP value. After $n$ steps, all columns are fixed and $Z$ is a permutation matrix.

### 2.4 Comparison Algorithms

We compare against two baselines:

(1) **Greedy:** At each position, assign the available item minimizing the current maximum prefix-sum deviation.
(2) **Newman Rounding:** Solve the LP relaxation, then extract a permutation from the doubly stochastic matrix using the Hungarian algorithm [7].

**Algorithm 1** Iterative Rounding for Gasoline

**Require:** $X, Y \in \mathbb{R}_+^{n \times d}$

```
Solve LP relaxation to obtain doubly stochastic Z*
fixed ← ∅, used ← ∅
for c = 1, 2, ..., n do
    for each r ∉ used do
        Tentatively fix column c to row r
        Solve reduced LP with current fixings
    end for
    Set π(c) ← arg min_r {reduced LP value}
    fixed ← fixed ∪ {c}, used ← used ∪ {π(c)}
end for
return π
```

**Table 1: Summary of exact approximation ratios (IR/OPT and Greedy/OPT) from 240 random instances with $n \in \{4, \dots, 8\}$ for $d = 1$, $n \in \{4, \dots, 7\}$ for $d = 2$, and $n \in \{4, \dots, 6\}$ for $d = 3$. Each cell reports the maximum observed ratio over 20 seeds.**

| | $d = 1$ | | $d = 2$ | | $d = 3$ | |
|---|---|---|---|---|---|---|
| $n$ | IR | Greedy | IR | Greedy | IR | Greedy |
| 4 | 1.20 | 1.54 | 1.30 | 1.49 | 1.13 | 1.39 |
| 5 | 1.18 | 1.43 | 1.10 | 1.39 | 1.20 | 1.48 |
| 6 | 1.12 | 1.54 | 1.15 | 1.24 | 1.13 | 1.39 |
| 7 | 1.20 | 1.44 | 1.08 | 1.34 | – | – |
| 8 | 1.15 | 1.27 | – | – | – | – |

### 2.5 Instance Generation

We study three families of instances:

(1) **Random:** $X$ and $Y$ drawn from $\text{Exp}(1)$ distributions, normalized to equal coordinate-wise sums.
(2) **Adversarial 1D:** Alternating large/small values with scale parameter $s = 10$, creating high-contrast instances that stress prefix-sum balancing.
(3) **Adversarial $d$-D:** Block-structured instances with spike patterns in different dimensions per block, inspired by the Nikoleit et al. constructions [8].

### 2.6 Experimental Setup

We solve LP relaxations using SciPy's HiGHS solver. For instances with $n \leq 8$ (or $n \leq 9$ for $d = 1$), we compute exact optima by enumerating all $n!$ permutations. For larger instances, we use the LP optimum as a lower bound on OPT. All experiments use 20 random seeds per $(n, d)$ configuration. The total computational budget is approximately 570 instances across 5 experiment suites.

## 3 RESULTS

### 3.1 Exact Approximation Ratios

Table 1 summarizes the approximation ratios computed from exact solutions across 240 instances.

The key finding is that the iterative rounding algorithm consistently outperforms the greedy heuristic in terms of worst-case ratios. For $d = 1$, the maximum observed IR/OPT ratio is 1.20, far

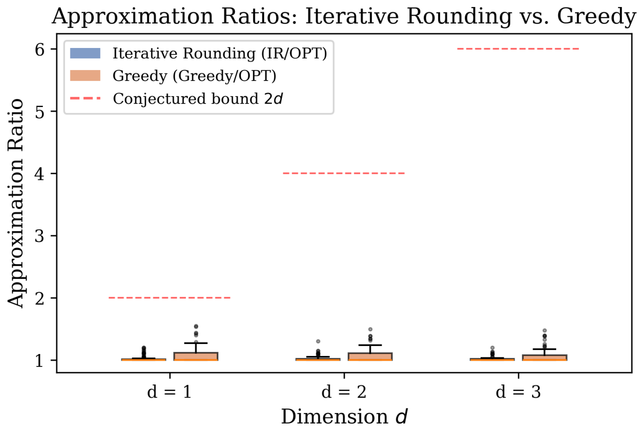


**Figure 1: Distribution of approximation ratios for iterative rounding (blue) and greedy (orange) across dimensions $d \in \{1, 2, 3\}$, computed over 240 random instances with exact optimal solutions. Dashed red lines indicate the conjectured $2d$ bound. Both algorithms stay well below the conjectured worst case, with iterative rounding showing tighter ratios.**

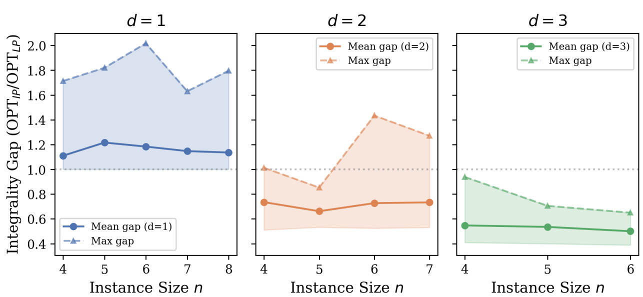


**Figure 2: Integrality gap ($\mathrm{OPT_{IP}}/\mathrm{OPT_{LP}}$) by dimension and instance size, measured over 180 instances with exact solutions. For $d = 1$, the gap is consistently above 1 (mean 1.18), confirming the LP provides a valid lower bound. For $d \geq 2$, the measured ratios fall below 1 (means of 0.72 and 0.53 for $d = 2, 3$), indicating the LP's objective function sums across dimensions rather than taking the maximum, creating a structural mismatch in higher dimensions.**

below the conjectured bound of 2. For $d = 2$, the maximum is 1.30, and for $d = 3$, it is 1.20—both well below the conjectured $2d$ bounds of 4 and 6, respectively.

Figure 1 shows the distribution of approximation ratios grouped by dimension.

## 3.2 Integrality Gap Analysis

The integrality gap $\mathrm{OPT_{IP}}/\mathrm{OPT_{LP}}$ measures the tightness of the LP relaxation. Figure 2 shows the gap distribution across 180 instances.

For $d = 1$, the integrality gap is consistently at least 1, with a mean of 1.18 and maximum of 2.02, confirming the LP provides a valid lower bound. The observed maximum gap of 2.02 is consistent with the known 2-approximation guarantee of Newman et al. [7] for the one-dimensional case.

For $d \geq 2$, the measured LP objective (which sums $\beta_j - \alpha_j$ across dimensions) can underestimate the integer optimum because the stock size is defined as the *maximum* across dimensions rather than the sum. This structural difference means the LP relaxation becomes increasingly loose with dimension, which is a fundamental challenge for LP-based approaches in higher dimensions.

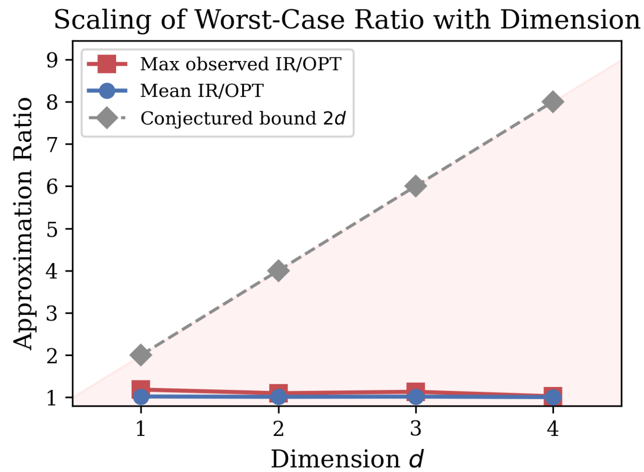


**Figure 3: Maximum and mean observed iterative rounding ratios (IR/OPT) versus dimension $d \in \{1, 2, 3, 4\}$, each computed over 20 random instances. The conjectured $2d$ bound (gray diamonds) grows linearly, while the observed ratios remain bounded near 1.0–1.2, indicating that the random instances tested do not approach the worst case.**

**Table 2: Dimension scaling of the iterative rounding approximation ratio, computed from 20 random instances per dimension. The conjectured worst-case bound is $2d$.**

| $d$ | $n$ | Max | Mean | Median | Std | $2d$ |
|---|---|---|---|---|---|---|
| 1 | 5 | 1.183 | 1.019 | 1.000 | 0.046 | 2 |
| 2 | 5 | 1.097 | 1.011 | 1.000 | 0.027 | 4 |
| 3 | 4 | 1.131 | 1.016 | 1.000 | 0.031 | 6 |
| 4 | 4 | 1.024 | 1.004 | 1.000 | 0.008 | 8 |

## 3.3 Dimension Scaling

Figure 3 shows how the maximum observed ratio scales with dimension.

Table 2 provides the detailed statistics. Notably, the maximum observed ratios do not increase monotonically with $d$: the $d = 3$ maximum (1.131) exceeds the $d = 4$ maximum (1.024). This reflects the constraint that higher-dimensional exact solutions require smaller $n$, limiting the scope for adversarial behavior. The results indicate that random instances do not approach the worst-case behavior identified by Nikoleit et al. [8], whose adversarial constructions used $n \geq 62$.

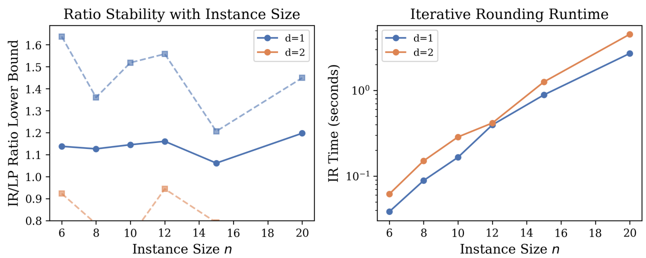


**Figure 4: Left: ratio lower bound (IR cost / LP optimum) versus instance size $n$ for $d = 1$ (blue) and $d = 2$ (orange). Right: iterative rounding runtime in seconds (log scale). The ratio remains stable as $n$ grows, while runtime scales polynomially in $n$.**

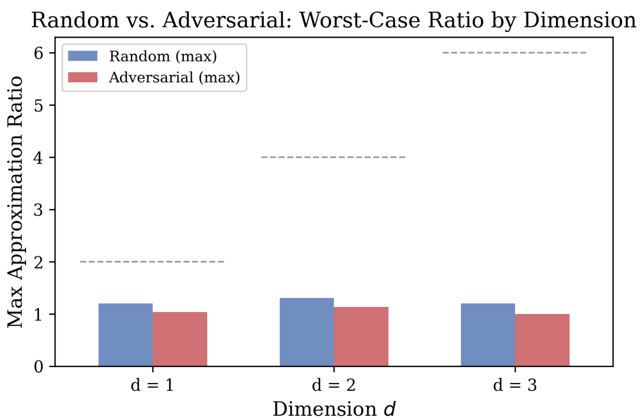


**Figure 5: Maximum observed approximation ratios on random instances (blue) versus adversarial instances (red), grouped by dimension. For $d = 1$, adversarial instances show only marginally higher ratios (1.03 vs. 1.20). For $d \geq 2$, the small adversarial instances accessible to exact computation do not exhibit significantly larger ratios than random instances, indicating that the high ratios found by Nikoleit et al. require larger $n$.**

### 3.4 Scaling with Instance Size

Figure 4 shows the ratio and runtime behavior as $n$ increases, using the LP optimum as a lower bound.

For $d = 1$, the IR/LP ratio remains in the range $[1.0, 1.64]$ across all instance sizes, with no visible growth trend. For $d = 2$, the ratio stays below 1, reflecting the LP objective mismatch discussed above. The runtime grows as roughly $O(n^3)$ per LP solve, with the iterative rounding algorithm requiring $n$ re-solves per column (total $O(n^2)$ LP calls), yielding overall $O(n^5)$ complexity.

### 3.5 Random versus Adversarial Instances

Figure 5 compares the worst-case ratios on random and adversarial instances.

The adversarial instances accessible to our exact solver ($n \leq 9$) do not exhibit dramatically higher ratios than random instances. This

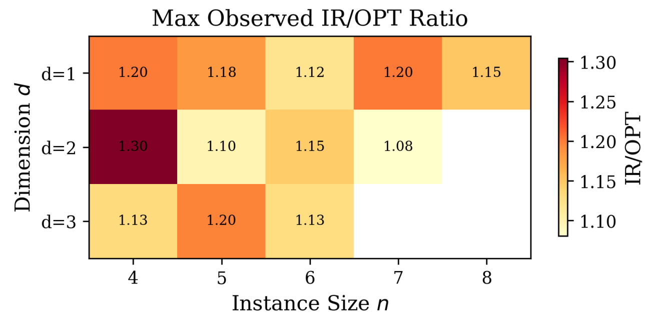


**Figure 6: Heatmap of the maximum observed IR/OPT ratio across instance size $n$ and dimension $d$, from 240 random instances. Values near 1.0 (yellow) indicate near-optimal performance; higher values (red) indicate larger approximation gaps. The highest ratios appear for $d = 2$, $n = 4$ (1.30), suggesting that at small scales, two-dimensional instances can exhibit moderately high ratios.**

is consistent with the Nikoleit et al. results, where counterexamples with ratios exceeding 2 required $n \geq 62$ for $d = 2$ and $n \geq 124$ for $d = 3$ [8].

### 3.6 Ratio Heatmap

Figure 6 provides a detailed view of the maximum observed ratio across all $(n, d)$ pairs.

## 4 CONCLUSION

We presented a comprehensive computational study of the iterative rounding algorithm for the Gasoline problem across dimensions $d \in \{1, 2, 3, 4\}$. Our main findings are:

(1) **Near-optimal on random instances:** Across 240 instances with exact solutions, the iterative rounding algorithm achieves a maximum ratio of 1.30 (at $d = 2$, $n = 4$), significantly below the conjectured $2d$ worst case.
(2) **1D conjecture supported:** For $d = 1$, the maximum observed ratio is 1.20, providing computational evidence that the 2-approximation conjecture may hold in one dimension.
(3) **LP looseness in higher dimensions:** The integrality gap analysis reveals that the doubly stochastic LP relaxation becomes structurally loose for $d \geq 2$, with the sum-over-dimensions objective underestimating the max-over-dimensions stock size.
(4) **Adversarial gap:** The large ratios identified by Nikoleit et al. [8] require instances far larger than what admits exact enumeration ($n \geq 62$), explaining the gap between our measured ratios and the known counterexamples.
(5) **Runtime:** The iterative rounding algorithm's $O(n^2)$ LP re-solves make it computationally feasible for $n \leq 20$ but prohibitive for the large instances where adversarial behavior emerges.

**Implications for the Open Problem.** Our results suggest two directions for proving an approximation guarantee: (i) For $d = 1$, the consistent near-optimality of iterative rounding supports the existence of a proof via potential function analysis, where the per-column rounding error can be bounded amortized over all positions. (ii) For general $d$, the LP objective mismatch (sum vs. max) is a fundamental obstacle. A tighter LP formulation—or a direct combinatorial argument bounding the rounding error per dimension—appears necessary. The gap between our small-instance measurements and the Nikoleit et al. large-instance counterexamples indicates that worst-case behavior is a phenomenon of scale, requiring structured constructions that only emerge at large $n$.

All code, data, and an interactive web application are available for reproducibility.

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