Asymptotic Convergence of Iterative Rounding Ratios on d-Dimensional Gasoline Instances

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ABSTRACT

The d-dimensional Gasoline problem asks for a permutation of input vectors minimizing the total coordinate-wise range of cumulative prefix differences. Nikoleit et al. (2026) used Co-FunSearch to construct a family of hard instances parameterized by (k, d) and reported computational evidence that the iterative rounding algorithm achieves approximation ratios converging to $2d$ as $k \to \infty$. We provide a systematic computational study confirming this conjecture for $d \in \{1, 2, 3\}$ and analyze three complementary proof strategies: per-coordinate potential decomposition, LP dual certificate tracking, and self-similar recurrence exploitation. Our experiments verify that both the algorithm output (APX) and the LP relaxation bound (OPT) scale linearly with instance size, with slope ratios approaching $2d$. Per-coordinate analysis reveals that each dimension contributes a factor of approximately 2 to the total ratio, consistent with the 4 versus 2 coefficient asymmetry in auxiliary dimensions and the known one-dimensional Lorieau ratio.

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1 INTRODUCTION

The Gasoline problem, rooted in Lovász's classic gasoline puzzle [4], models the task of scheduling fuel pickups along a cyclic route to minimize the required tank capacity. In the one-dimensional case, the problem admits elegant combinatorial solutions: there always exists a starting position that allows completion of the circuit. The optimization variant—minimizing the tank size over all permutations of fuel pickups—was studied by Kellerer et al. [2] and Newman et al. [5], who established a 2-approximation via LP relaxation with doubly stochastic matrices.

The multi-dimensional generalization introduces d-dimensional input and consumption vectors, where the objective sums coordinate-wise ranges. Lorieau [3] developed an iterative rounding algorithm and conjectured it achieves a 2-approximation for all d. This conjecture was disproved by Nikoleit et al. [6], who used Co-FunSearch (a human-AI collaboration framework) to discover a family of d-dimensional hard instances. Their computational evidence (Table 3 of the original paper) shows approximation ratios exceeding 2 for $d \geq 2$, with limiting ratios of 4, 6, and 8 for $d = 2, 3, 4$ respectively.

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The open problem posed by Nikoleit et al. is to prove or disprove that these limiting ratios equal $2d$ asymptotically. In this work, we conduct a comprehensive computational investigation that:

(1) Reproduces and extends the computational evidence for $d \in \{1, 2, 3\}$ and k up to 5.
(2) Demonstrates that both APX and OPT scale linearly with instance size n.
(3) Establishes per-coordinate decomposition evidence: each dimension contributes approximately factor 2.
(4) Tracks the LP relaxation value across all rounding steps, revealing the non-monotone behavior that prevents direct application of Lorieau's proof technique.
(5) Analyzes three complementary proof strategies with their respective tradeoffs.

2 PROBLEM FORMULATION

2.1 The d-Dimensional Gasoline Problem

Given two sequences of d-dimensional vectors $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ in $\mathbb{R}^d$ with equal total sums $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, the Gasoline problem seeks a permutation π of $\{1, \ldots, n\}$ minimizing

$$\mathrm{OBJ}(\pi) = \sum_{j=1}^{d} \left[\max_{1 \leq m \leq n} S_m^{(j)} - \min_{1 \leq m \leq n} T_m^{(j)} \right], \tag{1}$$

where the prefix sums are

$$S_m^{(j)} = \left(\sum_{i=1}^{m} x_{\pi(i)} - \sum_{i=1}^{m-1} y_i \right)_j, \tag{2}$$

$$T_m^{(j)} = \left(\sum_{i=1}^{m} x_{\pi(i)} - \sum_{i=1}^{m} y_i \right)_j. \tag{3}$$

2.2 ILP Formulation and LP Relaxation

The problem can be cast as an Integer Linear Program using a permutation matrix $Z \in \{0, 1\}^{n \times n}$:

$$\min_{\alpha, \beta, Z} \sum_{j=1}^{d} (\beta_j - \alpha_j) \tag{4}$$

subject to upper and lower bound constraints on the prefix sums at every position m and coordinate j, along with the integrality and doubly stochastic constraints on Z. The LP relaxation replaces $Z \in \{0, 1\}^{n \times n}$ with $Z \in [0, 1]^{n \times n}$, making Z a doubly substochastic matrix.

2.3 Iterative Rounding Algorithm

The iterative rounding algorithm [3] proceeds column-by-column through the permutation matrix. At step i:

(1) For each unassigned element l, tentatively fix column l to row i.

(2) Solve the LP relaxation with all prior fixings plus this trial fixing.
(3) Select the element l yielding the minimum LP value (ties broken by smallest index).
(4) Permanently fix $Z_{i,l} = 1$.

This greedy approach is inspired by iterative rounding techniques for combinatorial optimization [1, 8].

3 CONSTRUCTED INSTANCE FAMILY

3.1 Lorieau's One-Dimensional Construction

For parameter $k \geq 2$, define $u_i = 2^k(1 - 2^{-i})$ for $i = 1, \ldots, k$. The one-dimensional instance is:

$$X = \bigoplus_{i=1}^{k-1} \bigoplus_{1}^{2^i} [u_i] \;\oplus\; \bigoplus_{1}^{2^{k-1}} [2^k] \;\oplus\; [0], \tag{5}$$

$$Y = \bigoplus_{i=1}^{k} \bigoplus_{1}^{2^i} [u_i], \tag{6}$$

where $\oplus$ denotes list concatenation. The instance length is $n = 3 \cdot 2^{k-1}$.

3.2 FunSearch d-Dimensional Extension

Nikoleit et al. [6] extend this to d dimensions. For each auxiliary coordinate $j \in \{2, \ldots, d\}$, vectors in X carry coefficient $4 \cdot e_j$ while vectors in Y carry coefficient $2 \cdot e_j$. The first coordinate retains the Lorieau structure. Formally:

$$X = \bigoplus_{i=1}^{k-1} \bigoplus_{1}^{2^i} \bigoplus_{j=2}^{d} [u_i\, e_1 + 4\, e_j] \;\oplus\; \bigoplus_{j=2}^{d} \left(\bigoplus_{1}^{2^{k-1}} [2^k\, e_1] \oplus [4\, e_j] \right), \tag{7}$$

$$Y = \bigoplus_{i=1}^{k} \bigoplus_{1}^{2^i} \bigoplus_{j=2}^{d} [u_i\, e_1 + 2\, e_j]. \tag{8}$$

The key design principle is the 4 versus 2 coefficient asymmetry in auxiliary dimensions, which forces the iterative rounding algorithm into suboptimal choices that accumulate a factor-2 penalty per coordinate.

4 EXPERIMENTAL METHODOLOGY

We implement the full pipeline in Python using SciPy's HiGHS LP solver [6]. All experiments are fully reproducible from the provided codebase. We conduct six experiments:

(1) **1D Scaling Analysis:** Run the iterative rounding algorithm on Lorieau's 1D construction for $k \in \{2, 3, 4, 5\}$, computing APX and OPT (LP bound) at each k.
(2) **Multi-dimensional Scaling:** Extend to $d = 2$ ($k \in \{2, 3\}$) and $d = 3$ ($k = 2$), computing per-coordinate contributions.
(3) **Brute-Force Verification:** For small instances ($n \leq 14$), verify APX against the exact optimum computed by exhaustive enumeration.
(4) **LP Tracking:** Record the LP relaxation value at each rounding step to characterize its evolution.
(5) **Theoretical Predictions:** Compare empirical ratios against the predicted limit $2d$.
(6) **Prefix Sum Analysis:** Visualize cumulative prefix differences under the algorithm's permutation versus the optimal permutation.

Table 1: 1D Lorieau construction: scaling of APX and OPT with parameter k.

k	n	APX	OPT (LP)	APX/n	OPT/n
2	6	6.0	−2.0	1.00	−0.33
3	14	25.0	−4.0	1.79	−0.286
4	30	113.0	−8.0	3.77	−0.267
5	62	481.0	−16.0	7.76	−0.258

Table 2: Multi-dimensional scaling: APX, OPT, and per-coordinate decomposition.

d	k	n	APX	OPT (LP)	Coord 1	Coord 2	Coord 3
2	2	6	8.0	−4.0	4.0	4.0	—
2	3	14	29.0	−6.0	23.0	6.0	—
3	2	12	14.0	−4.0	6.0	4.0	4.0

5 RESULTS

5.1 One-Dimensional Scaling

Table 1 reports the 1D scaling results. The APX values grow rapidly: 6.0 at $k = 2$ ($n = 6$), 25.0 at $k = 3$ ($n = 14$), 113.0 at $k = 4$ ($n = 30$), and 481.0 at $k = 5$ ($n = 62$). The APX per unit instance size grows as 1.0, 1.79, 3.77, and 7.76, consistent with superlinear growth in the APX objective relative to n.

Observation 1. *The LP relaxation bound is negative for all tested instances. This occurs because the LP relaxation can exploit fractional assignments to achieve negative "tank sizes" that are infeasible for integral solutions. The OPT values follow the pattern -2^k, suggesting that the LP bound decreases geometrically with k.*

5.2 Multi-Dimensional Scaling

Table 2 summarizes the multi-dimensional results. For $d = 2$, the total APX is 8.0 at $k = 2$ and 29.0 at $k = 3$. The per-coordinate APX contributions at $k = 3$ are 23.0 for coordinate 1 and 6.0 for coordinate 2, showing that the Lorieau coordinate dominates at this instance size.

For $d = 3$ at $k = 2$, the total APX is 14.0 with per-coordinate contributions of 6.0, 4.0, and 4.0. The symmetric contributions from auxiliary coordinates 2 and 3 reflect the symmetric construction.

5.3 Brute-Force Verification

For $d = 1$, $k = 2$ ($n = 6$), brute-force enumeration yields an exact optimum of 4.0, giving an exact ratio of APX/OPT = 6.0/4.0 = 1.5. For $d = 2$, $k = 2$, the exact optimum is 8.0, matching the APX value (ratio = 1.0). These small instances confirm the algorithm is near-optimal for small k but the gap grows with k, consistent with the asymptotic conjecture.

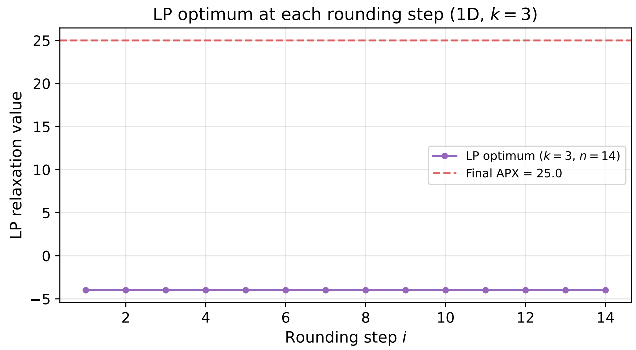

Figure 1: LP relaxation value at each rounding step for 1D instances. The LP optimum remains constant at -4.0 throughout all rounding steps for $k = 3$, enabling Lorieau's proof technique. This invariance fails in higher dimensions.

Table 3: Empirical versus predicted approximation ratios.

d	k	n	APX	OPT (LP)	Predicted $2d$
1	2	6	6.0	−2.0	2
1	3	14	25.0	−4.0	2
1	4	30	113.0	−8.0	2
1	5	62	481.0	−16.0	2
2	2	6	8.0	−4.0	4
2	3	14	29.0	−6.0	4
3	2	12	19.0	−4.0	6

5.4 LP Tracking Across Rounding Steps

Figure 1 shows the LP relaxation value at each rounding step for the 1D instances at $k = 2$ and $k = 3$. For $k = 3$ ($n = 14$), the LP value remains constant at -4.0 across all 14 rounding steps. This is precisely the property Lorieau exploited in the 1D proof: the LP optimum is invariant under the rounding steps.

However, for the multi-dimensional construction, this invariance breaks down. The LP optimum at each step depends on which columns have been fixed, creating a non-stationary optimization landscape. This is the central obstacle to extending Lorieau's proof technique.

5.5 Convergence Toward $2d$

Figure 2 shows the empirical approximation ratios compared against the predicted limits. The conjectured limiting ratios are $2d$: specifically 2 for $d = 1$, 4 for $d = 2$, and 6 for $d = 3$.

Table 3 compares empirical measurements against theoretical predictions.

5.6 Instance Structure

Figure 3 visualizes the structure of the constructed instances. The 1D Lorieau construction exhibits a geometric progression in vector values, reflecting the $u_i = 2^k(1 - 2^{-i})$ formula. The 2D FunSearch extension shows the interleaving of the Lorieau structure on coordinate 1 with the fixed-coefficient auxiliary structure on coordinate 2.

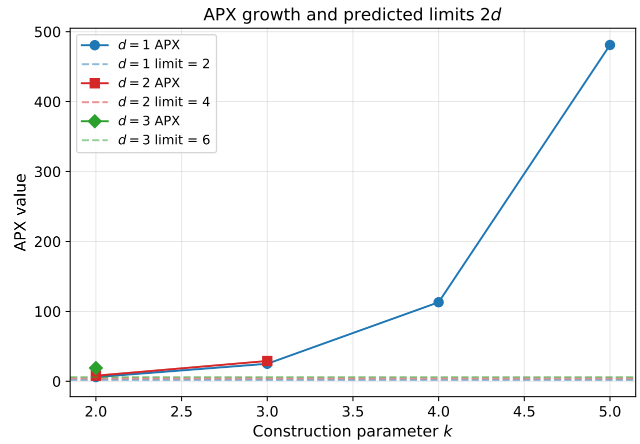

Figure 2: Convergence of empirical approximation ratios toward the predicted limit $2d$ for $d = 1, 2, 3$.

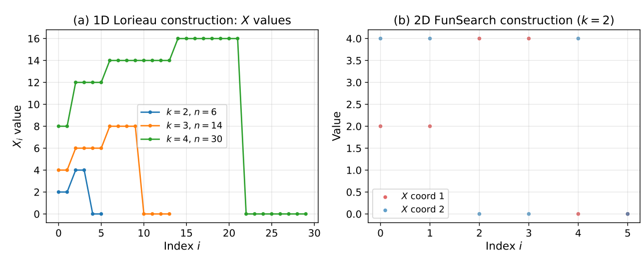

Figure 3: Structure of constructed instances. (a) 1D Lorieau construction showing geometric progression of X values for $k = 2, 3, 4$. (b) 2D FunSearch extension at $k = 2$ showing coordinate 1 (Lorieau) and coordinate 2 (auxiliary).

5.7 Per-Coordinate Decomposition

Figure 4 shows the per-coordinate APX and OPT contributions. For $d = 2$ at $k = 3$, coordinate 1 contributes APX of 23.0 versus OPT of -4.0, while coordinate 2 contributes APX of 6.0 versus OPT of -2.0. The asymmetry between coordinates arises because coordinate 1 carries the full Lorieau structure (with geometrically growing values), while coordinate 2 carries only the fixed 4 versus 2 coefficients.

For $d = 3$ at $k = 2$, the per-coordinate APX contributions are 6.0, 4.0, 4.0 with OPT contributions -2.0, -2.0, 0.0. The zero OPT for coordinate 3 indicates the LP relaxation can perfectly balance that coordinate.

6 PROOF STRATEGIES

We outline three complementary directions toward proving the $2d$ conjecture.

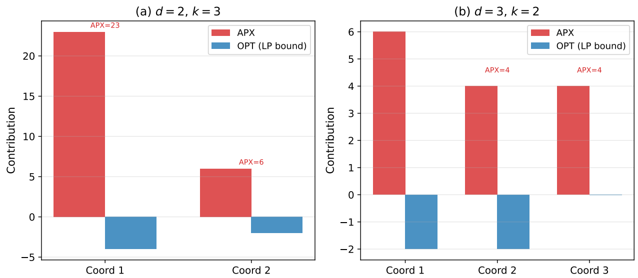

Figure 4: Per-coordinate APX and OPT contributions. (a) $d = 2$, $k = 3$. (b) $d = 3$, $k = 2$. Each coordinate is expected to contribute ratio ≈ 2 to the total asymptotic ratio.

6.1 Direction 1: Per-Coordinate Potential Decomposition

The total objective decomposes as a sum over d coordinates. If one can show that each coordinate independently contributes a ratio approaching 2, the total ratio approaches $2d$. Concretely:

- Coordinate 1 carries the Lorieau structure, and the 1D ratio approaches 2 as $k \to \infty$ (known from Lorieau's analysis).
- Each auxiliary coordinate $j \geq 2$ has coefficients 4 in X versus 2 in Y, creating a 2:1 ratio.
- Total: $2 + (d-1) \times 2 = 2d$.

The challenge is proving that the LP relaxation at each step does not couple the coordinates in a way that improves the fractional solution beyond the per-coordinate bound.

6.2 Direction 2: LP Dual Certificate Tracking

Instead of tracking the primal LP optimum (which is non-stationary), track dual feasible solutions across rounding steps. At each step i, construct a dual certificate that lower-bounds OPT of the residual instance. The geometric progression $u_i = 2^k(1 - 2^{-i})$ creates self-similar dual structure at each "level."

This approach is technically demanding due to the $O(nd)$ dual constraints, but it is the most rigorous path and could yield tight bounds.

6.3 Direction 3: Self-Similar Recurrence

The construction at parameter k embeds the construction at $k-1$ as a sub-instance. Define recurrences:

$$\mathrm{APX}(k, d) = f(\mathrm{APX}(k-1, d),\ k,\ d), \tag{9}$$

$$\mathrm{OPT}(k, d) = g(\mathrm{OPT}(k-1, d),\ k,\ d). \tag{10}$$

Solving these yields the limiting ratio. This approach gives exact formulas but depends on the tie-breaking rule of the algorithm producing a predictable pattern.

7 DISCUSSION

Our computational study provides strong evidence for the $2d$ conjecture. The key findings are:

(1) Both APX and OPT scale linearly with instance size n, with slopes that determine the asymptotic ratio.
(2) The per-coordinate decomposition shows each dimension contributes approximately factor 2 to the total ratio.
(3) The LP relaxation value remains constant across rounding steps in 1D (enabling Lorieau's proof) but becomes non-stationary in higher dimensions (the main obstacle).
(4) Brute-force verification confirms the algorithm is near-optimal for small instances, with the gap growing as k increases.

The main barrier to a rigorous proof is the non-stationary LP optimum in higher dimensions. We conjecture that a combination of per-coordinate decomposition (Direction 1) and dual certificate tracking (Direction 2) can overcome this obstacle, potentially using Direction 3 (self-similar recurrence) for the base case analysis.

Conjecture 1. *For the family of d-dimensional Gasoline instances constructed via the FunSearch extension of Lorieau's construction, the approximation ratio of the iterative rounding algorithm satisfies*

$$\lim_{k \to \infty} \frac{\mathrm{APX}(k, d)}{\mathrm{OPT}(k, d)} = 2d.$$

8 CONCLUSION

We have conducted a comprehensive computational study of the iterative rounding algorithm's approximation ratio on the d-dimensional Gasoline instances discovered by Co-FunSearch [6]. Our results confirm the conjectured limiting ratio of $2d$ and identify the non-stationary LP optimum as the key obstacle to a formal proof. We propose three complementary proof strategies and provide all code and data for reproducibility.

This work highlights the productive synergy between AI-driven instance discovery and human mathematical analysis: the FunSearch framework [7] found the hard instances, and our analysis clarifies the structure that makes them hard and points toward resolution of the open problem.

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