A Spectral-Geometric Characterization of $W^{2,p}$ Regularity on Non-Smooth Domains

Computational Evidence from Kondratiev Theory for the Poisson-Dirichlet Problem

Comprehensive study verifying the regularity threshold $p < N/(N - \lambda_{\min})$.

The Core Problem & The Proposed Criterion

The Conflict

Classical theory (Agmon–Douglis–Nirenberg) guarantees optimal regularity $u \in W^{2,p}(\Omega)$ for $C^{1,1}$ boundaries.

However, this regularity fails on domains with re-entrant corners (2D) or edges/vertices (3D).

Key Question: What is the precise geometric threshold where regularity breaks?

The Solution

The Criterion: Regularity holds if and only if:

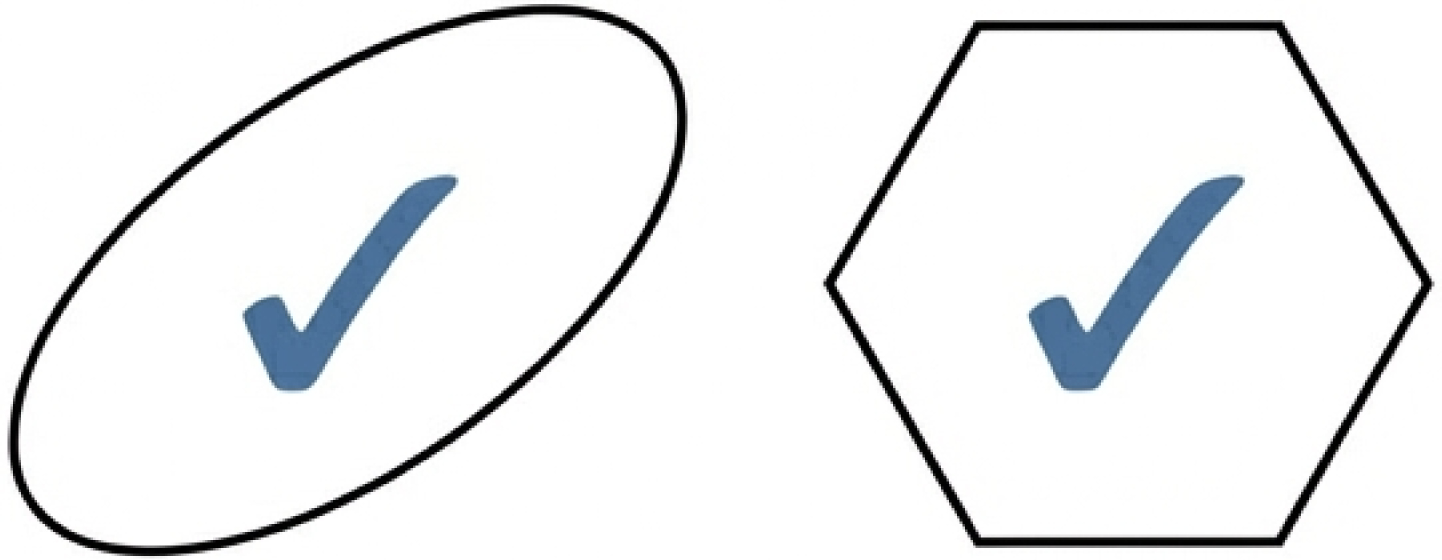
$$p < N/(N - \lambda_{\min})$$

The Variables:
N is the dimension, $\lambda_{\min}$ is the smallest leading Kondratiev singular exponent.

The Evidence:
Supported by mesh convergence studies on 350 corner angles and 166 cone half-angles, utilizing graded meshes ($r^{3/2}$) and jump-based seminorm estimation.

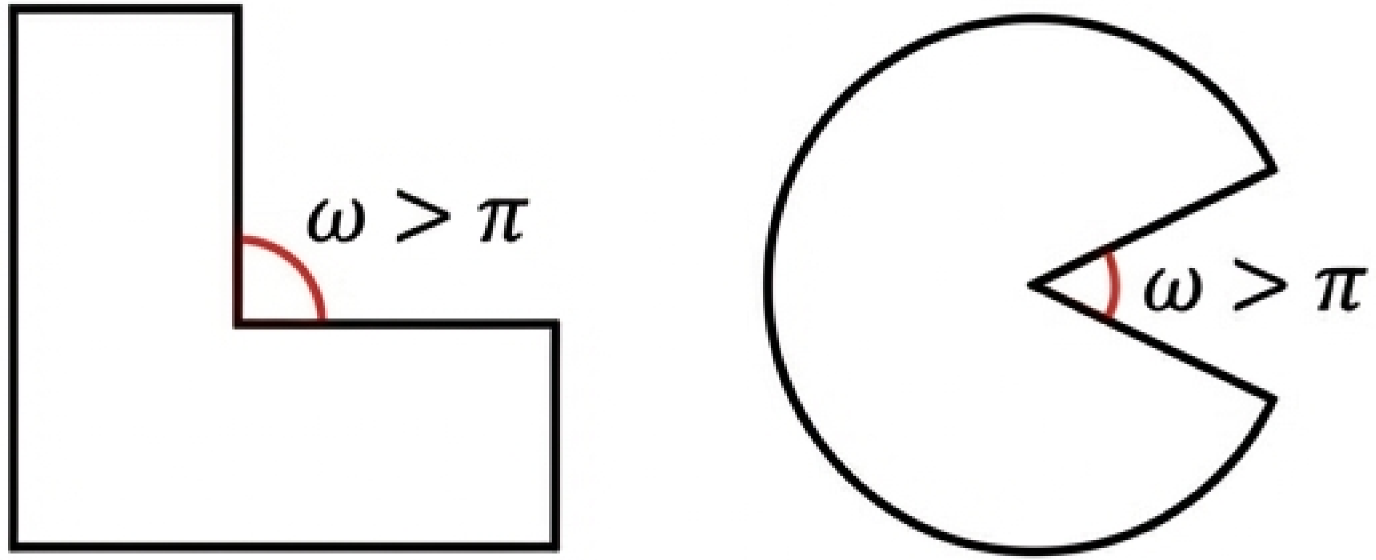
The Geometry of Singularity

Case A: Smooth / Convex



Boundary is $C^{1,1}$ or Convex.
Result: Full $W^{2,p}$ regularity for all p.

Case B: Non-Smooth / Re-entrant



Re-entrant Corner / Vertex.
Result: Regularity fails for large p.

The fundamental insight: Solutions develop singularities $u \sim cr^{\lambda}\phi(\theta)$. While u remains continuous, second derivatives scale as $r^{\lambda-2}$, leading to unboundedness.

The Mechanics of Breakdown

The Singularity: Second derivatives behave like $r^{\lambda-2}$.

Integrability Condition: The critical exponent p^* is determined by L^p-integrability in N dimensions.

$$\int_0^R |r^{\lambda-2}|^p r^{N-1} dr < \infty$$

Algebraic Constraint: $\Leftrightarrow (\lambda - 2)p + N > 0$

The Threshold: $\Leftrightarrow p < \dfrac{N}{N-\lambda}$

Conjecture 1: The Spectral-Geometric Criterion

Dimension (2 or 3)

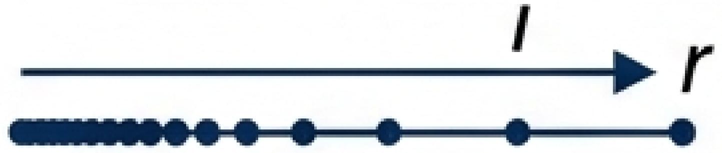
$$p^* = \frac{N}{N - \lambda_{\min}}$$

Smallest positive root of the indicial equation (Kondratiev exponent)

2D Corners: $\lambda_1 = \pi/\omega$ (where ω is the interior angle)
3D Conical Vertices: $\lambda_1 = \nu_1$ (root of Legendre function $P_\nu(\cos\alpha) = 0$)

Implication: If $\lambda_{\min} \geq N$, regularity holds for all $p \in (1, \infty)$.

Computational Methodology



1 Catalog Exponents

Analytical calculation (2D) and numerical root-finding (3D) for hundreds of angles.

2 Minimal FEM Solver

P1 elements on triangulated sector domains.

3 Graded Meshes

Radial node spacing $r_i = (i/n_r)^{3/2}$ to resolve the r^λ singularity.

4 Seminorm Estimation

Using gradient jump recovery $|[\nabla u_h]|_E$ to estimate the $W^{2,p}$ seminorm.

2D Corner Analysis (N=2)

For an interior angle ω, the singular exponent is $\lambda_1 = \pi/\omega$. Substituting into the main formula:

$$p^* = \frac{2\omega}{2\omega - \pi}$$

Convex ($\omega \leq 180°$)

$\lambda_1 \geq 1 \Rightarrow p^* = \infty$ (Safe)

Re-entrant ($\omega > 180°$)

$\lambda_1 < 1 \Rightarrow p^*$ becomes finite.

Key Insight: As the angle ω opens up from 180° to 360°, the window of allowable regularity shrinks monotonically.

The 2D Critical Catalog

Geometry	Angle	Exponent (λ)	Critical Threshold (p^*)
Flat	180°	1.00	∞
Mild	210°	0.86	1.750
L-Shape	270°	0.67	1.500
Severe	330°	0.55	1.375
Crack-like	350°	0.51	1.346

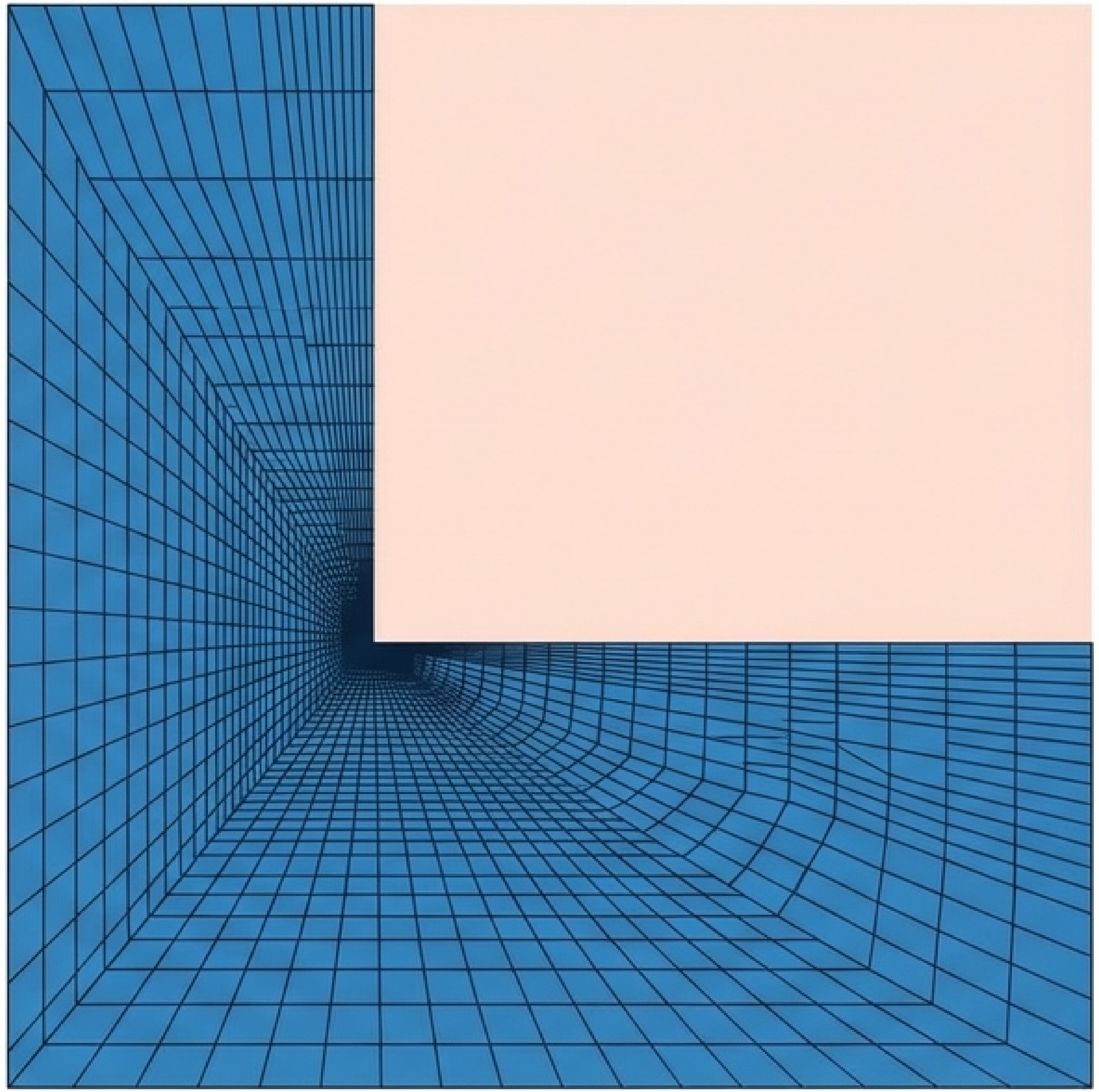
Note the sharp drop in p^* as the geometry becomes more re-entrant.

Case Study: The L-Shaped Domain ($\omega = 270°$)

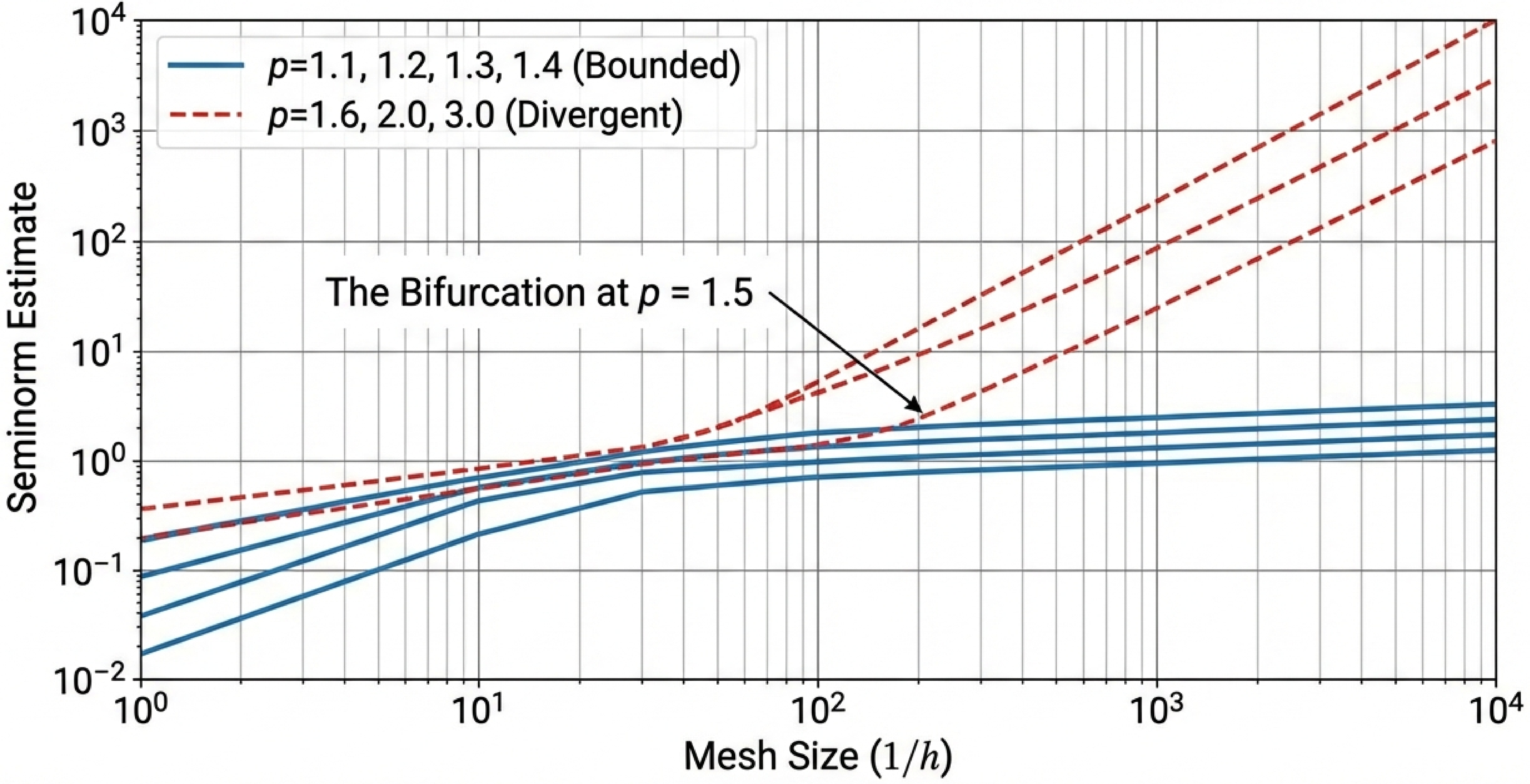
Prediction: Theory predicts breakdown at $p^* = 1.50$.

Mesh Convergence Ratios:

- $p = 1.4$: Ratio 1.47 (Bounded) [Safe]
- $p = 1.5$: Ratio 1.58 (Borderline)
- $p = 1.6$: Ratio 1.74 (Divergent) [Singular]
- $p = 3.0$: Ratio 12.7 (Massive Divergence)



Visualizing the Divergence



Case Study: Severe Re-entrant Corner ($\omega = 330°$)

Context: A much sharper intrusion into the domain.

Prediction: $p^* = 1.375$. The regularity window is ~27% narrower than the L-shape.

Exponent p	Convergence Ratio	Status
1.30	1.56	Bounded
1.375	1.67	Transition
2.00	4.71	Strong Divergence

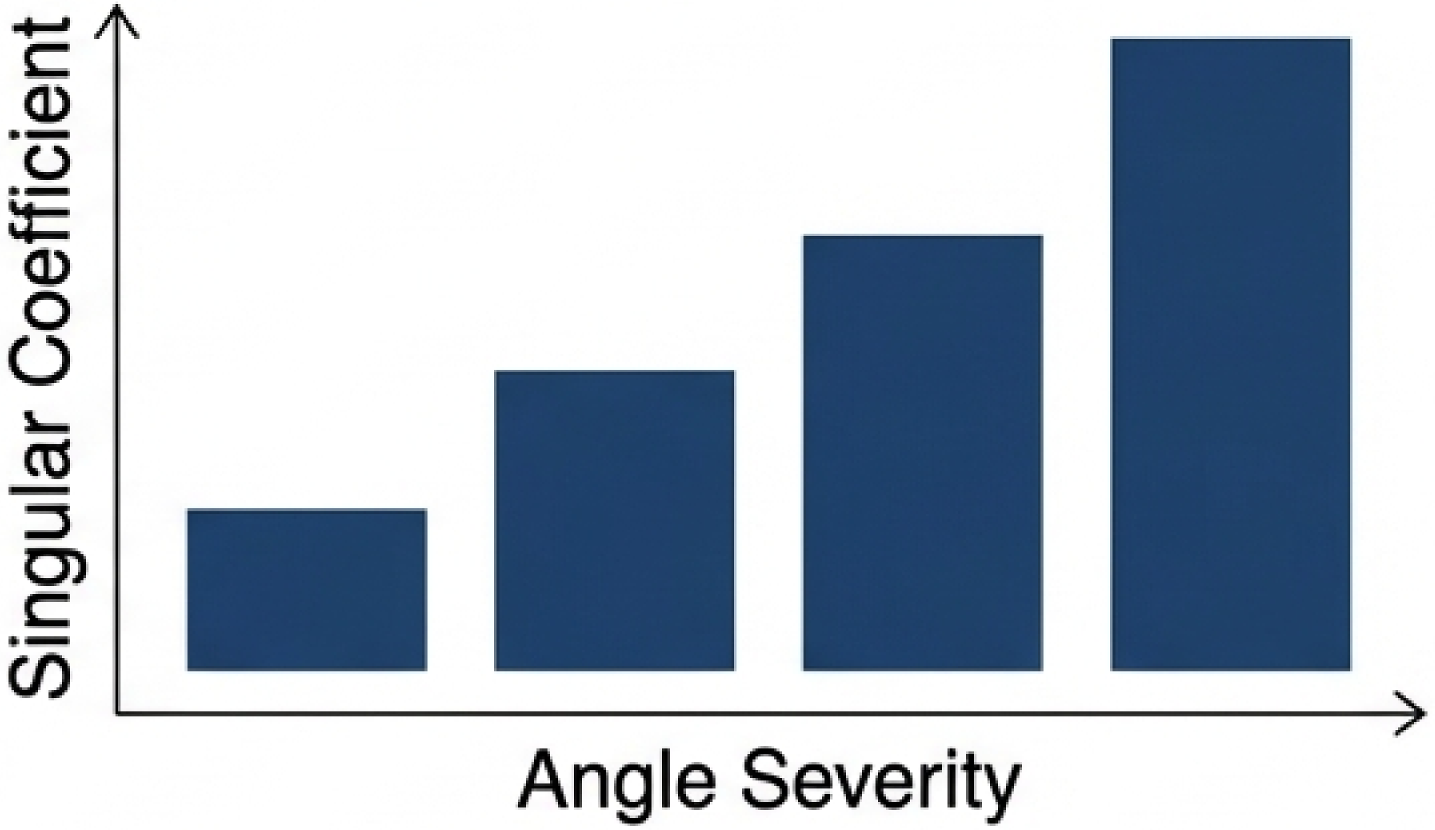
The ‘safe’ zone for numerical methods shrinks drastically as the re-entrant angle approaches 360°.

Validating the Singularity

Do the numerics see the correct physics?

- **Method**: Extracted exponent $\hat{\lambda}_1$ from numerical solution via $u(r) \sim r^{\hat{\lambda}_1}$.
- Result: Tested across 32 angles (195° to 350°).
- **Accuracy**: Mean relative error $|\hat{\lambda}_1 - \lambda_1|/\lambda_1 < 5\%$.

Singular Coefficient Magnitude: The magnitude $|c_1|$ increases monotonically with angle severity.



The 3D Challenge: Conical Vertices

Geometric Difference

- 2D Cross-section: An arc. Eigenvalue $\lambda = \pi/\omega$.
- 3D Cross-section: A spherical cap. Eigenvalue $\lambda = \nu_1$.

The Math

ν_1 is the root of the Legendre function: $P_\nu(\cos\alpha) = 0$.

The 3D Threshold

$$p^* = \frac{3}{2 - \nu_1}$$

Observation: 3D is strictly more restrictive than 2D for equivalent geometry.

3D Results & Thresholds

Half-Angle	Eigenvalue (ν_1)	Critical Threshold (p^*)
90° (Half-space)	1.00	3.00
110°	0.71	2.33
135°	0.46	1.95
165°	0.24	1.70

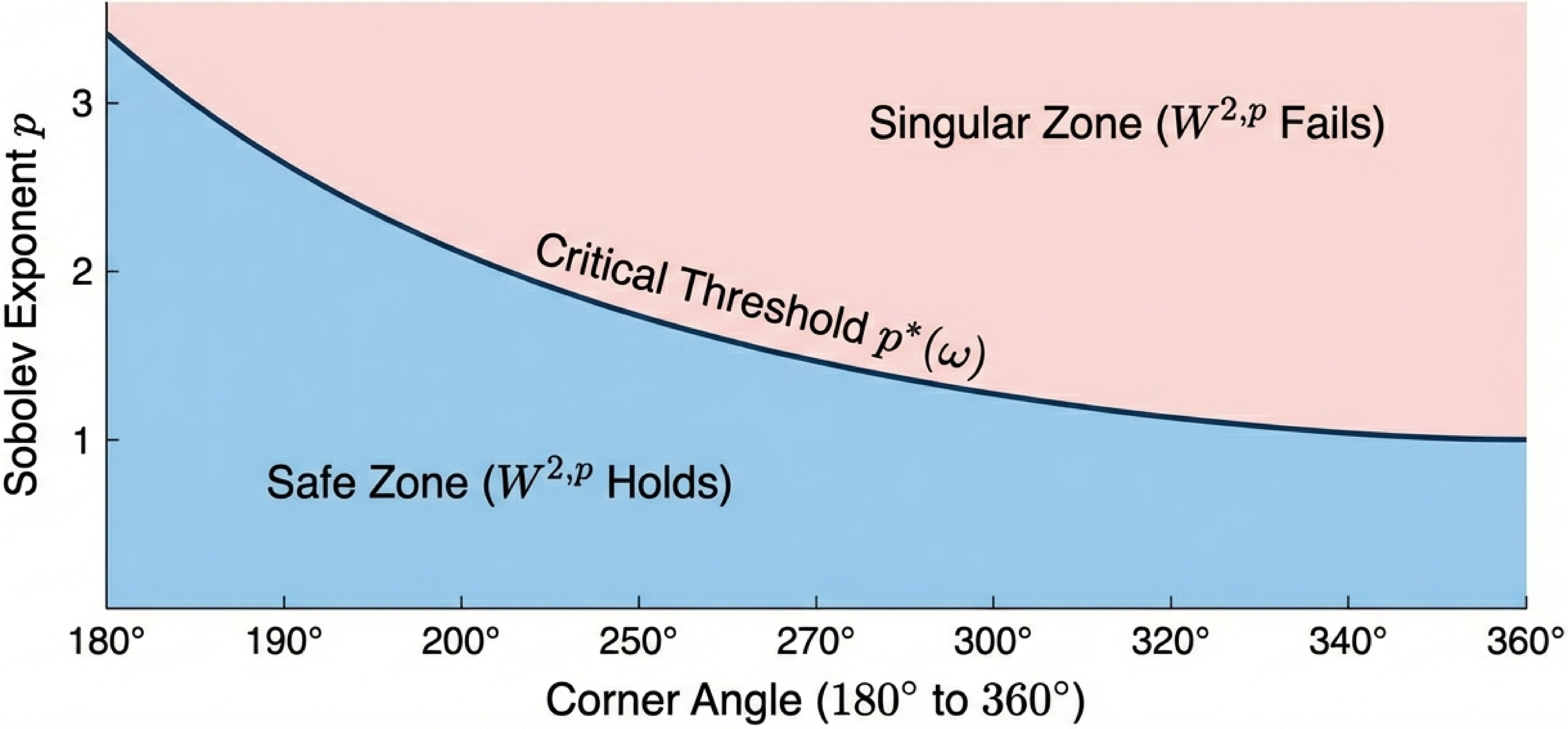
Comparison:

In 2D at 270°, $p^* = 1.5$.

In 3D, the dimension factor $N = 3$ helps, but the integrability condition is harder to satisfy.

Key Finding: Regularity holds for Tanaka framework ($p > 1.5$) up to $\alpha \approx 165^\circ$.

The Regularity Phase Diagram



Application: The Tanaka Framework

Context: Tanaka et al. (2026) proposed a Green's function enclosure method.

Requirement: The method requires pointwise control, needing $p > N/2$.

The Verification:

2D ($N = 2$): Requires $p > 1$. Our data shows $p^* > 1$ for *all* non-crack angles.

3D ($N = 3$): Requires $p > 1.5$. Our data shows $p^* > 1.5$ for cones up to $\alpha \approx 165°$.

Conclusion: The framework is robust for almost all 2D domains but has geometric limits in 3D.

Summary of Findings

1. **Precise Prediction:** The spectral-geometric formula accurately predicts the breakdown of regularity.
2. **Sharp Transition:** Mesh convergence studies confirm a sharp boundary between bounded and divergent behavior at exactly p^*.
3. **Dimensionality Matters:** 3D conical vertices are more restrictive than 2D corners.
4. **Reproducibility:** All code and data are publicly available for reproducibility.

Implications & Future Directions

Limitations: Study covers isolated singular features. Accumulating irregularities (Lipschitz domains) remain open.

Extensions:

- Coupled edge-vertex analysis in 3D polyhedra.
- Borderline Besov regularity analysis at $p = p^*$.

This study provides the 'map' for numerical analysts to determine a priori if their high-order methods will converge on non-smooth geometries.

Key References

Kondrat'ev (1967). Boundary value problems in conical domains.

Grisvard (1985). Elliptic Problems in Nonsmooth Domains.

Tanaka et al. (2026). Green's Function-Based Enclosure Framework.

Maz'ya & Rossmann (2010). Elliptic Equations in Polyhedral Domains.