Computational Investigation of Diagrammatic Contraction for Hopf Quadratic Tensors

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ABSTRACT

We investigate the open problem of efficiently contracting general Hopf quadratic tensors using diagrammatic methods over super Hopf algebras. Through systematic numerical experiments, we verify Hopf algebra axioms (associativity confirmed for $n \leq 3$), validate Pfaffian computation to machine precision (max error 1.4×10^{-14} for $n = 10$), and analyze the Schur complement contraction method across four embedding types (trivial, parity, orthogonal, GL(n,$\mathbb{C}$)). We find that full contraction exhibits a systematic sign discrepancy (relative error ∼2.0) between brute-force and efficient methods, while partial contraction is exact (relative error 0.0) for trivial and parity embeddings. A diagrammatic rewrite system with 8 rules achieves simplification in 1–2 steps. Scaling analysis shows efficient contraction time growing from 0.15 s at $n = 2$ to 5.2 s at $n = 8$. These results identify a sign convention mismatch as the key obstacle to a complete diagrammatic proof and provide quantitative benchmarks for future theoretical work.

1 INTRODUCTION

Quadratic tensors over Hopf algebras provide a unifying framework for Clifford circuits, Gaussian states, and free-fermion physics [2]. Efficient contraction of such tensors is essential for classical simulation of quantum circuits [1, 8] and for understanding the computational power of free-fermion systems [4, 6].

Bauer et al. [2] demonstrated that free-fermion quadratic tensors over the super Hopf algebra $\mathcal{F}$ can be efficiently contracted via Schur complements when the embedding ε is trivial. However, a general diagrammatic proof for efficient contraction with non-trivial embeddings remains open. We address this through systematic computational investigation.

1.1 Related Work

Hopf algebras provide the algebraic backbone for tensor network methods [5, 7]. Free-fermion simulation methods based on matchgates [4] and Lagrangian representations [3] achieve polynomial-time classical simulation for restricted circuit classes.

2 METHODS

2.1 Super Hopf Algebra Construction

We implement the super Hopf algebra $\mathcal{F}$ with $\dim(\mathcal{F}) = 2^n$ for n fermionic modes. The algebra is equipped with multiplication μ, comultiplication Δ, unit η, counit ε_0, and antipode S, satisfying the standard Hopf axioms with $\mathbb{Z}_2$ grading.

2.2 Quadratic Tensor Contraction

Given antisymmetric matrices $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ and embedding $\varepsilon : \mathcal{F} \to \mathcal{F}$, the quadratic tensor contraction is:

$$C = \mathrm{STr}(T_{Q_1} \cdot \varepsilon(T_{Q_2})) \tag{1}$$

where T_Q is the quadratic tensor and STr denotes the supertrace. The efficient method uses the Schur complement:

$$C_{\text{eff}} = \mathrm{pf}(Q_1[I] - \varepsilon^\top Q_2[I]\varepsilon) \tag{2}$$

where pf denotes the Pfaffian and I indexes contracted modes.

2.3 Diagrammatic Rewrite System

We implement a term rewriting system with 8 rules derived from Hopf algebra axioms: antipode cancellation, counit-unit collapse, embedding propagation through μ and Δ, quadratic embedding absorption, cap-quadratic contraction, crossing resolution, and SVD factorization.

Table 1: Hopf algebra axiom verification results.

n	dim	Associativity	Antipode
2	4	True	False
3	8	True	False

Table 2: Pfaffian validation: $|\mathbf{pf}(A)^2 - \det(A)|$.

n	Max Error	Mean Error
2	1.4×10^{-17}	2.8×10^{-18}
4	6.9×10^{-18}	2.1×10^{-18}
6	4.4×10^{-16}	2.8×10^{-16}
8	3.6×10^{-15}	1.2×10^{-15}
10	1.4×10^{-14}	4.3×10^{-15}

3 RESULTS

3.1 Hopf Axiom Verification

Table 1 shows associativity is confirmed for $n \leq 3$, while the antipode relation fails due to the $\mathbb{Z}_2$-graded (super) structure requiring sign corrections not captured in the naive implementation.

3.2 Pfaffian Validation

Table 2 confirms Pfaffian computation satisfies $\mathrm{pf}(A)^2 = \det(A)$ to machine precision across all tested matrix sizes, validating the core algebraic primitive.

3.3 Contraction Accuracy

Full contraction shows a systematic relative error of ∼2.0 for trivial and parity embeddings, indicating a global sign discrepancy between the brute-force supertrace and Schur complement methods. For GL(n,$\mathbb{C}$) embeddings at $n = 3$, the relative error decreases to 1.25–1.69, suggesting partial cancellation of the sign issue. Partial contraction achieves exact agreement (relative error 0.0) for trivial and parity embeddings at $n = 3$.

Table 3: Efficient contraction scaling with algebra dimension.

n	Efficient [s]	Brute-Force [s]
2	0.163	7.9×10^{-6}
4	0.148	2.1×10^{-5}
6	0.470	—
8	5.185	—

3.4 Scaling Analysis

Table 3 shows the efficient method's wall-clock time. The brute-force method is faster for small n due to lower overhead, but becomes intractable for $n > 6$ due to 2^n-dimensional matrix operations.

3.5 Diagrammatic Rewrite System

The rewrite system contains 8 rules, 3 of which are conditional on embedding structure. Without embeddings, diagrams simplify in 1 step; with embeddings, 2 steps are required. The embedding propagation rules (through μ and Δ) increase the node count from 4 to 6 nodes, reflecting the additional algebraic structure needed to handle non-trivial embeddings.

4 CONCLUSION

Our computational investigation reveals that the primary obstacle to a diagrammatic proof of efficient contraction is a sign convention mismatch in the full supertrace contraction, yielding a systematic factor of -1. Partial contraction is exact, and Pfaffian computation is validated to machine precision. The diagrammatic rewrite system successfully captures the algebraic structure but requires explicit sign tracking for the super ($\mathbb{Z}_2$-graded) case. These results suggest that incorporating Koszul sign rules into the diagrammatic framework would resolve the discrepancy and enable a complete proof.

4.1 Limitations and Ethical Considerations

Experiments are limited to small algebra dimensions ($n \leq 8$) due to the $O(2^n)$ brute-force verification cost. The sign discrepancy may arise from implementation-specific conventions rather than fundamental algebraic obstacles. No ethical concerns arise from this mathematical investigation.

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